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Uncertain SEIAR model for COVID-19 cases in China

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Abstract
The Susceptible-Exposed-Infectious-Asymptomatic-Removed (SEIAR) epidemic model is one of most frequently used epidemic models. As an application of uncertain differential equations to epidemiology, an uncertain SEIAR model is derived which considers the human uncertainty factors during the spread of an epidemic. The parameters in the uncertain epidemic model are estimated with the numbers of COVID-19 cases in China, and a prediction to the possible numbers of active cases is made based on the estimates.

Keywords Uncertainty theory · Uncertain differential equation · Uncertain SEIAR model · COVID-19 · Parameter estimation

1 Introduction

Epidemics have always been a threat to human health. To describe and predict the spread of an epidemic, various models have been built such as the SIS model, SIR model, SEIR model and SEIAR model. Since the establishment of stochastic differential equation theory in 1950s, stochastic epidemic models have been investigated for the reasons of indeterminate factors during the spread process of an epidemic. For example, Gray et al. (2011) presented a stochastic SIS model, and gave some conditions for extinction and persistence of the disease. Ji et al. (2012) discussed a stochastic SIR model, and proved the stability conditions of disease-free equilibrium. Artalejo et al. (2015) presented a stochastic SEIR model, and studied the evolution of the epidemic before its extinction. As we know, stochastic differential equations are applicable to dynamic systems with random factors rather than human uncertainty, so it is questionable whether they can properly describe epidemic systems which are heavily affected by human behaviors.

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As a branch of mathematics for modelling human uncertainty, the uncertainty theory was founded by Liu (2007) and perfected by Liu (2009). Within the framework of uncertainty theory, the concept of uncertain differential equation was proposed by Liu (2008) to describe dynamic systems with human uncertainty. Chen and Liu (2010) gave a sufficient condition for an uncertain differential equation to have a unique solution, and Yao et al. (2013) proved some stability theorems about uncertain differential equations. The structure of the solution of an uncertain differential equation was found by Yao and Chen (2013), based on which various numerical methods have been designed to solve uncertain differential equations, such as Yang and Ralescu (2015), Gao (2016), and Zhang et al. (2017). In order to estimate the parameters in uncertain differential equations based on observed data, Yao and Liu (2020) presented the method of moments, which was extended to the generalized method of moments by Liu (2020). In addition, the least squares estimation and the maximum likelihood estimation for uncertain differential equations were presented by Sheng et al. (2020) and Liu and Liu (2020), respectively.

As an application of uncertain differential equations to finance, uncertain finance theory was extended during the past years. For example, Liu (2009) assumed the price of a stock follows a lognormal uncertainty distribution, and derived the European option pricing formulas. Chen and Gao (2013) described the short-term interest rate with uncertain differential equations, and investigated the pricing problems of zero-coupon bonds. Uncertain differential equations have also been applied to optimal control (Zhu 2010), game theory (Yang and Gao 2015), population growth model (Sheng et al. 2017) and pharmacokinetics model (Liu and Yang 2020), etc.

The investigation of uncertain epidemic models was initialized by Li et al. (2017), where an uncertain SIS model was constructed and the disease-free equilibrium was discussed. After that, the uncertain SIS model was generalized to the cases with standard incidence and demography (Fang et al. 2018) and with nonlinear incidence and demography (Li and Teng 2019). However, applications of these SIS models are limited because the groups of exposed individuals and recovered individuals with potential immunity are not taken into consideration in these models. In this paper, we aim to build a more general epidemic model by means of uncertain differential equations that is called uncertain SEIAR model. Groups of susceptible individuals, exposed individuals, symptomatically infected individuals, asymptomatically infected individuals, and removed individuals are all considered in such a model, and its application to the COVID-19 cases in China is also provided. The rest of this paper is organized as follows. Section 2 introduces some basic concepts related to uncertain differential equations. Section 3 derives the uncertain SEIAR model, and gives the uncertain SEIR model and the uncertain SIR model as degenerated forms of uncertain SEIAR model. Section 4 estimates the parameters in the uncertain epidemic model by using the numbers of COVID-19 cases in China, and Sect. 5 predicts the possible numbers of COVID-19 active cases in China by means of the uncertain epidemic model with those estimated parameters. Finally, some conclusion are made in Sect. 6.

2 Preliminary

The uncertain measure $\mathcal{M}$ is a set function from a measurable space $(\Gamma, \mathcal{L})$ to the interval $[0, 1]$ which satisfies the normality, duality, subadditivity and product axioms. An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

Definition 1 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

An uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(e, \sigma)$. A normal uncertainty distribution is called standard if $e = 0$ and $\sigma = 1$.

Definition 2 (Liu 2007) Let ξ be an uncertain variable, and let k be a positive integer. Then the k-th moment of ξ is

$$E[\xi^k] = \int_0^{+\infty} \mathcal{M}\left\{\xi^k \geq x\right\} \mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\left\{\xi^k \leq x\right\} \mathrm{d}x$$

provided that at least one of the two integrals is finite.

The first and second moments of a standard normal uncertain variable $\mathcal{N}(0, 1)$ are 0 and 1, respectively.

Definition 3 (Liu 2009) An uncertain process C_t is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(ii) C_t has stationary and independent increments;
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable $\mathcal{N}(0, t)$.

Definition 4 (Liu 2009) Let X_t be an uncertain process, and let C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Definition 5 (Liu 2008) Suppose that C_t is a Liu process, and f and g are two measurable functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation.

The integral form of the uncertain differential equation (1) is

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s.$$

3 Uncertain SEIAR model

In an SEIAR model, the population is divided into 5 groups, namely, susceptible individuals, exposed individuals, symptomatically infected individuals, asymptomatically infected individuals, and removed individuals (recovered or dead). Susceptible individuals will become exposed individuals after contact with symptomatically infected individuals or asymptomatically infected individuals; Exposed individuals will possibly become symptomatically infected individuals or asymptomatically infected individuals; Symptomatically infected individuals and asymptomatically infected individuals will be removed in some proportions; Removed individuals will not become susceptible individuals again. For simplicity, the numbers of individuals in these 5 groups at the time t are denoted by S_t, E_t, I_t, A_t and R_t in the uncertain SEIAR model, respectively.

3.1 Equation of susceptible individuals

Let β_{1t} denote the contact rate between a susceptible individual and a symptomatically infected individual, and β_{2t} denote the contact rate between a susceptible individual and an asymptomatically infected individual. Considering the human uncertainty during the contact process, we assume

$$\beta_{1t} = \beta_I + \sigma_1 \cdot \text{“Noise”}$$

where β_I is a nonnegative number, and “Noise” is a normal uncertain variable $\mathcal{N}(0, 1)$. Representing the “Noise” by

$$\frac{C_{1,t+\Delta t} - C_{1t}}{\Delta t}$$

where C_{1t} is a Liu process, we further have

$$\beta_{1t} = \beta_I + \sigma_1 \frac{C_{1,t+\Delta t} - C_{1t}}{\Delta t}.$$

Similarly, the contact rate β_{2t} can be represented as

$$\beta_{2t} = \beta_A + \sigma_2 \frac{C_{2,t+\Delta t} - C_{2t}}{\Delta t}$$

where C_{2t} is a Liu process, and β_A and σ_2 are two nonnegative numbers. Then the increment of the number of susceptible individuals during an infinitesimal time interval $[t, t+\Delta t]$ is

$$\begin{aligned} S_{t+\Delta t} - S_t &= -(\beta_{1t} S_t I_t \Delta t + \beta_{2t} S_t A_t \Delta t) \\ &= -(\beta_I S_t I_t + \beta_A S_t A_t)\Delta t - \sigma_1 S_t I_t (C_{1,t+\Delta t} - C_{1t}) \\ &\quad - \sigma_2 S_t A_t (C_{2,t+\Delta t} - C_{2t}). \end{aligned}$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of susceptible individuals during such an interval is

$$\begin{aligned} S_t - S_0 &= \sum_{i=0}^{n-1} (S_{t_{i+1}} - S_{t_i}) \\ &= -\sum_{i=0}^{n-1} (\beta_I S_{t_i} I_{t_i} + \beta_A S_{t_i} A_{t_i})(t_{i+1} - t_i) - \sum_{i=0}^{n-1} \sigma_1 S_{t_i} I_{t_i} (C_{1t_{i+1}} - C_{1t_i}) \\ &\quad - \sum_{i=0}^{n-1} \sigma_2 S_{t_i} A_{t_i} (C_{2t_{i+1}} - C_{2t_i}) \\ &\to -\int_0^t (\beta_I S_\tau I_\tau + \beta_A S_\tau A_\tau) \mathrm{d}\tau - \sigma_1 \int_0^t S_\tau I_\tau \mathrm{d}C_{1\tau} - \sigma_2 \int_0^t S_\tau A_\tau \mathrm{d}C_{2\tau} \end{aligned}$$

as

$$\max_{0 \le i \le n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$\begin{aligned} \mathrm{d}S_t = &-(\beta_I S_t I_t + \beta_A S_t A_t)\mathrm{d}t - \sigma_1 S_t I_t \mathrm{d}C_{1t} \\ &-\sigma_2 S_t A_t \mathrm{d}C_{2t}. \end{aligned} \tag{2}$$

3.2 Equation of exposed individuals

Let ν_{1t} denote the rate that exposed individuals become symptomatically infected individuals, and ν_{2t} denote the rate that exposed individuals become asymptomatically infected individuals. Similar to the assumptions about the contact rates in Sect. 3.1, we assume that

$$\nu_{1t} = \nu_I + \sigma_3 \frac{C_{3,t+\Delta t} - C_{3t}}{\Delta t}$$

and

$$\nu_{2t} = \nu_A + \sigma_4 \frac{C_{4,t+\Delta t} - C_{4t}}{\Delta t}$$

where C_{3t} and C_{4t} are two Liu processes, and ν_I, ν_A, σ_3 and σ_4 are some nonnegative numbers. Then the increment of the number of exposed individuals during an infinitesimal time interval $[t, t + \Delta t]$ is

$$\begin{aligned} E_{t+\Delta t} - E_t &= (S_t - S_{t+\Delta t}) - (\nu_{1t} + \nu_{2t}) E_t \Delta t \\ &= (S_t - S_{t+\Delta t}) - (\nu_I + \nu_A) E_t \Delta t - \sigma_3 E_t (C_{3,t+\Delta t} - C_{3t}) \\ &\quad - \sigma_4 E_t (C_{4,t+\Delta t} - C_{4t}). \end{aligned}$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of exposed individuals during such an interval is

$$\begin{aligned} E_t - E_0 &= \sum_{i=0}^{n-1} (E_{t_{i+1}} - E_{t_i}) \\ &= S_0 - S_t - \sum_{i=0}^{n-1} (\nu_I + \nu_A) E_{t_i} (t_{i+1} - t_i) - \sum_{i=0}^{n-1} \sigma_3 E_{t_i} (C_{3t_{i+1}} - C_{3t_i}) \\ &\quad - \sum_{i=0}^{n-1} \sigma_4 E_{t_i} (C_{4t_{i+1}} - C_{4t_i}) \\ &\to \int_0^t (\beta_I S_\tau I_\tau + \beta_A S_\tau A_\tau) \mathrm{d}\tau + \sigma_1 \int_0^t S_\tau I_\tau \mathrm{d}C_{1\tau} + \sigma_2 \int_0^t S_\tau A_\tau \mathrm{d}C_{2\tau} \\ &\quad - (\nu_I + \nu_A) \int_0^t E_\tau \mathrm{d}\tau - \sigma_3 \int_0^t E_\tau \mathrm{d}C_{3\tau} - \sigma_4 \int_0^t E_\tau \mathrm{d}C_{4\tau} \end{aligned}$$

as

$$\max_{0 \le i \le n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$\begin{aligned} \mathrm{d}E_t = &(\beta_I S_t I_t + \beta_A S_t A_t - \nu_I E_t - \nu_A E_t) \mathrm{d}t + \sigma_1 S_t I_t \mathrm{d}C_{1t} + \sigma_2 S_t A_t \mathrm{d}C_{2t} \\ &- \sigma_3 E_t \mathrm{d}C_{3t} - \sigma_4 E_t \mathrm{d}C_{4t}. \end{aligned} \tag{3}$$

3.3 Equation of symptomatically infected individuals

Let ν_{1t} denote the rate that exposed individuals become symptomatically infected individuals, and μ_{1t} denote the removed (recovered or dead) rate of symptomatically infected individuals. As the assumption in Sect. 3.2, we have

$$\nu_{1t} = \nu_I + \sigma_3 \frac{C_{3,t+\Delta t} - C_{3t}}{\Delta t}.$$

Similarly, we assume that

$$\mu_{1t} = \mu_I + \sigma_5 \frac{C_{5,t+\Delta t} - C_{5t}}{\Delta t}$$

where C_{5t} is a Liu process, and μ_I and σ_5 are two nonnegative numbers. Then the increment of the number of symptomatically infected individuals during an infinitesimal time interval $[t, t + \Delta t]$ is

$$\begin{aligned} I_{t+\Delta t} - I_t &= \nu_{1t} E_t \Delta t - \mu_{1t} I_t \Delta t \\ &= (\nu_I E_t - \mu_I I_t)\Delta t + \sigma_3 E_t (C_{3,t+\Delta t} - C_{3t}) - \sigma_5 I_t (C_{5,t+\Delta t} - C_{5t}). \end{aligned}$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of symptomatically infected individuals during such an interval is

$$\begin{aligned} I_t - I_0 &= \sum_{i=0}^{n-1} (I_{t_{i+1}} - I_{t_i}) \\ &= \sum_{i=0}^{n-1} (\nu_I E_{t_i} - \mu_I I_{t_i})(t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_3 E_{t_i} (C_{3t_{i+1}} - C_{3t_i}) \\ &\quad - \sum_{i=0}^{n-1} \sigma_5 I_{t_i} (C_{5t_{i+1}} - C_{5t_i}) \\ &\to \int_0^t (\nu_I E_\tau - \mu_I I_\tau) \mathrm{d}\tau + \sigma_3 \int_0^t E_\tau \mathrm{d}C_{3\tau} - \sigma_5 \int_0^t I_\tau \mathrm{d}C_{5\tau} \end{aligned}$$

as

$$\max_{0 \le i \le n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$\mathrm{d}I_t = (\nu_I E_t - \mu_I I_t)\mathrm{d}t + \sigma_3 E_t \mathrm{d}C_{3t} - \sigma_5 I_t \mathrm{d}C_{5t}. \tag{4}$$

3.4 Equation of asymptomatically infected individuals

Let ν_{2t} denote the rate that exposed individuals become asymptomatically infected individuals, and μ_{2t} denote the removed (recovered or dead) rate of asymptomatically infected individuals. As the assumption in Sect. 3.2, we have

$$\nu_{2t} = \nu_A + \sigma_4 \frac{C_{4,t+\Delta t} - C_{4t}}{\Delta t}.$$

Similarly, we assume that

$$\mu_{2t} = \mu_A + \sigma_6 \frac{C_{6,t+\Delta t} - C_{6t}}{\Delta t}$$

where C_{6t} is a Liu process, and μ_A and σ_6 are two nonnegative numbers. Then the increment of the number of asymptomatically infected individuals during an infinitesimal time interval $[t, t + \Delta t]$ is

$$\begin{aligned} A_{t+\Delta t} - A_t &= \nu_{2t} E_t \Delta t - \mu_{2t} A_t \Delta t \\ &= (\nu_A E_t - \mu_A A_t)\Delta t + \sigma_4 E_t (C_{4,t+\Delta t} - C_{4t}) - \sigma_6 A_t (C_{6,t+\Delta t} - C_{6t}). \end{aligned}$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of asymptomatically infected individuals during such an interval is

$$\begin{aligned} A_t - A_0 &= \sum_{i=0}^{n-1} (A_{t_{i+1}} - A_{t_i}) \\ &= \sum_{i=0}^{n-1} (\nu_A E_{t_i} - \mu_A A_{t_i})(t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_4 E_{t_i} (C_{4t_{i+1}} - C_{4t_i}) \\ &\quad - \sum_{i=0}^{n-1} \sigma_6 A_{t_i} (C_{6t_{i+1}} - C_{6t_i}) \\ &\to \int_0^t (\nu_A E_\tau - \mu_A A_\tau) \mathrm{d}\tau + \sigma_4 \int_0^t E_\tau \mathrm{d}C_{4\tau} - \sigma_6 \int_0^t A_\tau \mathrm{d}C_{6\tau} \end{aligned}$$

as

$$\max_{0 \le i \le n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$\mathrm{d}A_t = (\nu_A E_t - \mu_A A_t)\mathrm{d}t + \sigma_4 E_t \mathrm{d}C_{4t} - \sigma_6 A_t \mathrm{d}C_{6t}. \tag{5}$$

3.5 Equation of removed individuals

Let μ_{1t} denote the removed rate of symptomatically infected individuals, and μ_{2t} denote the removed rate of asymptomatically infected individuals. As the assumptions in Sects. 3.3 and 3.4, we have

$$\mu_{1t} = \mu_I + \sigma_5 \frac{C_{5,t+\Delta t} - C_{5t}}{\Delta t}$$

and

$$\mu_{2t} = \mu_A + \sigma_6 \frac{C_{6,t+\Delta t} - C_{6t}}{\Delta t}.$$

The increment of the number of removed individuals during an infinitesimal time interval $[t, t + \Delta t]$ is

$$\begin{aligned} R_{t+\Delta t} - R_t &= \mu_{1t} I_t \Delta t + \mu_{2t} A_t \Delta t \\ &= (\mu_I I_t + \mu_A A_t)\Delta t + \sigma_5 I_t (C_{5,t+\Delta t} - C_{5t}) + \sigma_6 A_t (C_{6,t+\Delta t} - C_{6t}). \end{aligned}$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of removed individuals during such an interval is

$$\begin{aligned} R_t - R_0 &= \sum_{i=0}^{n-1} (R_{t_{i+1}} - R_{t_i}) \\ &= \sum_{i=0}^{n-1} (\mu_I I_{t_i} + \mu_A A_{t_i})(t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_5 I_{t_i} (C_{5t_{i+1}} - C_{5t_i}) \\ &\quad + \sum_{i=0}^{n-1} \sigma_6 A_{t_i} (C_{6t_{i+1}} - C_{6t_i}) \\ &\to \int_0^t (\mu_I I_\tau + \mu_A A_\tau)\mathrm{d}\tau + \sigma_5 \int_0^t I_\tau \mathrm{d}C_{5\tau} + \sigma_6 \int_0^t A_\tau \mathrm{d}C_{6\tau} \end{aligned}$$

as

$$\max_{0 \le i \le n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$\mathrm{d}R_t = (\mu_I I_t + \mu_A A_t)\mathrm{d}t + \sigma_5 I_t \mathrm{d}C_{5t} + \sigma_6 A_t \mathrm{d}C_{6t}. \tag{6}$$

3.6 Uncertain SEIAR model

Based on the equations in Sects. 3.1–3.5, we propose the uncertain SEIAR model

$$\begin{cases} dS_t = -(\beta_I S_t I_t + \beta_A S_t A_t)dt - \sigma_1 S_t I_t dC_{1t} - \sigma_2 S_t A_t dC_{2t} \\ dE_t = (\beta_I S_t I_t + \beta_A S_t A_t - \nu_I E_t - \nu_A E_t)dt + \sigma_1 S_t I_t dC_{1t} \\ \qquad +\sigma_2 S_t A_t dC_{2t} - \sigma_3 E_t dC_{3t} - \sigma_4 E_t dC_{4t} \\ dI_t = (\nu_I E_t - \mu_I I_t)dt + \sigma_3 E_t dC_{3t} - \sigma_5 I_t dC_{5t} \\ dA_t = (\nu_A E_t - \mu_A A_t)dt + \sigma_4 E_t dC_{4t} - \sigma_6 A_t dC_{6t} \\ dR_t = (\mu_I I_t + \mu_A A_t)dt + \sigma_5 I_t dC_{5t} + \sigma_6 A_t dC_{6t} \end{cases} \tag{7}$$

to describe an epidemic system with uncertain information, where S_t, E_t, I_t, A_t and R_t denote the numbers of susceptible individuals, exposed individuals, symptomatically infected individuals, asymptomatically infected individuals, and removed individuals, respectively, C_{it}, $i = 1, 2, \ldots, 6$ are Liu processes, β_I, β_A, ν_I, ν_A, μ_I, μ_A and σ_i, $i = 1, 2, \ldots, 6$ are some nonnegative numbers.

The uncertain SEIAR model (7) degenerates to an uncertain SEIR model if the group of asymptomatically infected individuals is neglected. Setting $A_t = 0$ and $\nu_A = \sigma_4 = 0$ in the model (7), we get the uncertain SEIR model

$$\begin{cases} dS_t = -\beta_I S_t I_t dt - \sigma_1 S_t I_t dC_{1t} \\ dE_t = (\beta_I S_t I_t - \nu_I E_t)dt + \sigma_1 S_t I_t dC_{1t} - \sigma_3 E_t dC_{3t} \\ dI_t = (\nu_I E_t - \mu_I I_t)dt + \sigma_3 E_t dC_{3t} - \sigma_5 I_t dC_{5t} \\ dR_t = \mu_I I_t dt + \sigma_5 I_t dC_{5t}. \end{cases} \tag{8}$$

Furthermore, the uncertain SEIR model (8) degenerates to an uncertain SIR model if the group of exposed individuals is neglected. In this case, all the exposed individuals are regarded as infected individuals. Replacing $\nu_I E_t dt + \sigma_3 E_t dC_{3t}$ with $\beta_I S_t I_t dt + \sigma_1 S_t I_t dC_{1t}$ in the model (8), we get the uncertain SIR model

$$\begin{cases} dS_t = -\beta_I S_t I_t dt - \sigma_1 S_t I_t dC_{1t} \\ dI_t = (\beta_I S_t I_t - \mu_I I_t)dt + \sigma_1 S_t I_t dC_{1t} - \sigma_5 I_t dC_{5t} \\ dR_t = \mu_I I_t dt + \sigma_5 I_t dC_{5t}. \end{cases} \tag{9}$$

4 Parameter estimation with COVID-19 cases

Based on the COVID-19 cases in Mainland China, we estimate the parameters in the uncertain epidemic models in this section. The focus of this research is on the number of active cases, so we perform the parameter estimation with respect to the uncertain SIR model for simplicity. Beside, we accept the following stipulation in order to further simplify the parameter estimation process.

Stipulation: The number of susceptible individuals is a constant that is 1.4 billion. According to National Bureau of Statistics of the People's Republic of China, the population of Mainland China is about 1.40005 billion, while the number of confirmed cases in mainland China is 82,052 as of April 11, 2020. Hence, we stipulate that the number of susceptible individuals is 1.4 billion for simplicity.

Following the above stipulation, the uncertain SIR model (9) is simplified to an IR model

$$\begin{cases} \mathrm{d}I_t = (\beta_I S_0 I_t - \mu_I I_t)\mathrm{d}t + \sigma_1 S_0 I_t \mathrm{d}C_{1t} - \sigma_5 I_t \mathrm{d}C_{5t} \\ \mathrm{d}R_t = \mu_I I_t \mathrm{d}t + \sigma_5 I_t \mathrm{d}C_{5t} \end{cases} \tag{10}$$

where $S_0 = 1.4$ billion is the number of susceptible individuals, I_t and R_t denote the numbers of symptomatically infected individuals and removed individuals, respectively, C_{1t} and C_{5t} are two Liu processes which are assumed to be independent for the purpose of parameter estimation, and β_I, μ_I, σ_1 and σ_5 are nonnegative parameters to be estimated.

The following parameter estimation process mainly follows the method of moments for uncertain differential equations by Yao and Liu (2020) and for multi-factor uncertain differential equations by Liu and Yang (2020). Consider the equation of removed individuals

$$\mathrm{d}R_t = \mu_I I_t \mathrm{d}t + \sigma_5 I_t \mathrm{d}C_{5t} \tag{11}$$

in the uncertain IR model (10). A Euler difference scheme of the uncertain differential equation (11) is

$$R_{t_{i+1}} - R_{t_i} = \mu_I I_{t_i}(t_{i+1} - t_i) + \sigma_5 I_{t_i}(C_{5t_{i+1}} - C_{5t_i})$$

which can be rewritten as

$$\frac{R_{t_{i+1}} - R_{t_i} - \mu_I I_{t_i}(t_{i+1} - t_i)}{\sigma_5 I_{t_i}(t_{i+1} - t_i)} = \frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}.$$

According to the definition of Liu process, the right expression

$$\frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}$$

is a standard normal uncertain variable $\mathcal{N}(0, 1)$. Hence, when R_{t_i} and I_{t_i} are assigned the numbers of closed cases and active cases on the i-th day, respectively, the left expression

$$\frac{R_{t_{i+1}} - R_{t_i} - \mu_I I_{t_i}(t_{i+1} - t_i)}{\sigma_5 I_{t_i}(t_{i+1} - t_i)}$$

can be regarded as samples of a standard normal uncertain variable $\mathcal{N}(0, 1)$. Note that the first and second moments of a standard normal uncertain variable $\mathcal{N}(0, 1)$ are 0 and 1, respectively. The estimates μ_I^* and σ_5^* of μ_i and σ_5 solve the system of equations

$$\begin{cases} \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \dfrac{R_{t_{i+1}} - R_{t_i} - \mu_I I_{t_i}(t_{i+1} - t_i)}{\sigma_5 I_{t_i}(t_{i+1} - t_i)} = 0 \\ \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \left(\dfrac{R_{t_{i+1}} - R_{t_i} - \mu_I I_{t_i}(t_{i+1} - t_i)}{\sigma_5 I_{t_i}(t_{i+1} - t_i)}\right)^2 = 1. \end{cases}$$

That is,

$$\mu_I^* = \frac{1}{n-1}\sum_{i=1}^{n-1} \frac{R_{t_{i+1}} - R_{t_i}}{I_{t_i}(t_{i+1} - t_i)}, \tag{12}$$

$$\sigma_5^* = \left(\frac{1}{n-1}\sum_{i=1}^{n-1} \left(\frac{R_{t_{i+1}} - R_{t_i}}{I_{t_i}(t_{i+1} - t_i)} - \mu_I^*\right)^2\right)^{1/2}. \tag{13}$$

Now consider the equation of symptomatically infected individuals

$$\mathrm{d}I_t = (\beta_I S_0 I_t - \mu_I I_t)\mathrm{d}t + \sigma_1 S_0 I_t \mathrm{d}C_{1t} - \sigma_5 I_t \mathrm{d}C_{5t}$$

in the uncertain IR model (10). Substituting μ_I and σ_5 with μ_I^* and σ_5^* that have been determined in Eqs. (12) and (13), we get

$$\mathrm{d}I_t = (\beta_I S_0 I_t - \mu_I^* I_t)\mathrm{d}t + \sigma_1 S_0 I_t \mathrm{d}C_{1t} - \sigma_5^* I_t \mathrm{d}C_{5t}. \tag{14}$$

A Euler difference scheme of the uncertain differential equation (14) is

$$\begin{aligned} I_{t_{i+1}} - I_{t_i} = &(\beta_I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i) + \sigma_1 S_0 I_{t_i}(C_{1t_{i+1}} - C_{1t_i}) \\ &-\sigma_5^* I_{t_i}(C_{5t_{i+1}} - C_{5t_i}) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{I_{t_{i+1}} - I_{t_i} - (\beta_I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i)}{(\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i})(t_{i+1} - t_i)} = &\frac{\sigma_1 S_0 I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \\ \frac{C_{1t_{i+1}} - C_{1t_i}}{t_{i+1} - t_i} - &\frac{\sigma_5^* I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}. \end{aligned}$$

It follows from the independence and definition of Liu processes C_{1t} and C_{5t} that

$$\frac{C_{1t_{i+1}} - C_{1t_i}}{t_{i+1} - t_i}$$

and

$$\frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}$$

Table 1 Numbers of active cases of COVID-19 in mainland China from February 12 to April 11, 2020. *Source* COVID-19 Cases Reports by the National Health Commission of the People's Republic of China National Health Commission of the People's Republic of China (2020)

52,526	55,748	56,873	57,416	57,934	58,016	57,805	56,303	54,965	53,284
51,606	49,824	47,672	45,604	43,258	39,919	37,414	35,329	32,652	30,004
27,433	25,352	23,784	22,177	20,533	19,016	17,721	16,145	14,831	13,526
12,094	10,734	9898	8976	8056	7263	6569	6013	5549	5120
4735	4287	3947	3460	3128	2691	2396	2161	2004	1863
1727	1562	1376	1299	1242	1190	1160	1116	1089	1138

are independent standard normal uncertain variables with a common uncertainty distribution $\mathcal{N}(0, 1)$. Then according to the operational law of uncertain variables, we have

$$\begin{aligned}&\frac{\sigma_1 S_0 I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \frac{C_{1t_{i+1}} - C_{1t_i}}{t_{i+1} - t_i} - \frac{\sigma_5^* I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i} \\ &\sim \mathcal{N}\left(0, \frac{\sigma_1 S_0 I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} + \frac{\sigma_5^* I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}}\right) = \mathcal{N}(0, 1).\end{aligned}$$

Hence, when I_{t_i} are assigned the numbers of active cases on the i-th day, the left expression

$$\frac{I_{t_{i+1}} - I_{t_i} - (\beta_I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i)}{(\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i})(t_{i+1} - t_i)}$$

can be regarded as samples of a standard normal uncertain variable $\mathcal{N}(0, 1)$. Then by means of the method of moments, the estimates β_I^* and σ_1^* of β_I and σ_1 solve the system of equations

$$\begin{cases} \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n-1} \dfrac{I_{t_{i+1}} - I_{t_i} - (\beta_I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i)}{(\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i})(t_{i+1} - t_i)} = 0 \\ \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n-1} \left(\dfrac{I_{t_{i+1}} - I_{t_i} - (\beta_I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i)}{(\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i})(t_{i+1} - t_i)}\right)^2 = 1. \end{cases}$$

That is,

$$\beta_I^* = \frac{1}{S_0}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} \frac{I_{t_{i+1}} - I_{t_i}}{I_{t_i}(t_{i+1} - t_i)} + \mu_I^*\right), \tag{15}$$

$$\sigma_1^* = \frac{1}{S_0}\left(\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{I_{t_{i+1}} - I_{t_i}}{I_{t_i}(t_{i+1} - t_i)} - (\beta_I^* S_0 - \mu_I^*)\right)^2\right)^{1/2} - \sigma_5^*\right). \tag{16}$$

Now, we estimate the parameters β_I, σ_1, μ_I and σ_5 in the uncertain IR model (10) based on the numbers of active cases (Table 1) and closed cases (Table 2) in

Table 2 Numbers of closed cases of COVID-19 in mainland China from February 12 to April 11, 2020. *Source* COVID-19 Cases Reports by the National Health Commission of the People's Republic of China National Health Commission of the People's Republic of China (2020)

7278	8103	9619	11,084	12,614	14,420	16,380	18,273	20,500	23,004
25,330	27,326	29,986	32,460	35,239	38,905	41,837	44,495	47,374	50,147
52,837	55,057	56,768	58,474	60,162	61,719	63,033	64,633	65,962	67,287
68,730	70,110	70,962	71,905	72,838	73,665	74,398	74,995	75,505	75,973
76,436	76,931	77,338	77,880	78,266	78,748	79,074	79,357	79,550	79,726
79,893	80,077	80,293	80,409	80,498	80,612	80,705	80,791	80,864	80,914

Mainland China from February 12 to April 11, 2020, which are released in the COVID-19 cases reports by National Health Commission of the People's Republic of China National Health Commission of the People's Republic of China (2020). Details about the computation procedure are given as below.

Step 1 Set $S_0 = 1.4 \times 10^9$ and $n = 60$.
Step 2 Set February 12 as the first day, February 13 as the second day, and so forth. That is, $t_i = i$, and I_{t_i} and R_{t_i} are the i-th numbers in Tables 1 and 2, respectively, for $i = 1, 2, \cdots, 60$. For example, $I_{t_3} = 56873$ and $R_{t_5} = 12614$.
Step 3 Compute μ_I^* based on Eq. (12).
Step 4 Compute σ_5^* based on Eq. (13).
Step 5 Compute β_I^* based on Eq. (15).
Step 6 Compute σ_1^* based on Eq. (16).

Following the above procedure, we get the estimates of the parameters β_I, σ_1, μ_I and σ_5 in the uncertain IR model (10) that are

$$\beta_I^* = 1.1745 \times 10^{-11}, \quad \sigma_1^* = 7.7866 \times 10^{-12}, \quad \mu_I^* = 0.0785, \quad \sigma_5^* = 0.0296.$$

Hence, the uncertain IR model for COVID-19 in Mainland China from February 12 to April 11, 2020 is

$$\begin{cases} \mathrm{d}I_t = -0.0620 \cdot I_t \mathrm{d}t + 0.0109 \cdot I_t \mathrm{d}C_{1t} - 0.0296 \cdot I_t \mathrm{d}C_{5t} \\ \mathrm{d}R_t = 0.0785 \cdot I_t \mathrm{d}t + 0.0296 \cdot I_t \mathrm{d}C_{5t}. \end{cases} \tag{17}$$

5 Prediction of the confirmed cases

Based on the uncertain IR model (10) with the estimated parameters, we predict the possible numbers of active cases (I_t) in this section.

Consider the equation of symptomatically infected individuals

$$\mathrm{d}I_t = (\beta_I^* S_0 I_t - \mu_I^* I_t)\mathrm{d}t + \sigma_1^* S_0 I_t \mathrm{d}C_{1t} - \sigma_5^* I_t \mathrm{d}C_{5t}$$

which has a solution

$$I_t = I_0 \exp\left((\beta_I^* S_0 - \mu_I^*)t + \sigma_1^* S_0 C_{1t} - \sigma_5^* C_{5t}\right).$$

Please note that its uncertainty distribution is

$$\Phi_t(x) = \mathcal{M}\{I_t \le x\} = \mathcal{M}\left\{\sigma_1^* S_0 C_{1t} - \sigma_5^* C_{5t} \le \ln x - (\beta_I^* S_0 - \mu_I^*)t - \ln I_0\right\}.$$

Since C_{1t} and C_{5t} are independent Liu processes with a common uncertainty distribution $\mathcal{N}(0, t)$, we have

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi((\beta_I^* S_0 - \mu_I^*)t + \ln I_0 - \ln x)}{\sqrt{3}(\sigma_1^* S_0 + \sigma_5^*)t}\right)\right)^{-1}. \tag{18}$$

A prediction interval of possible numbers that I_t may take at the time s with a confidence α is $[i_L, i_U]$ where (i_L, i_U) is the optimal solution of the optimization problem

$$\begin{cases} \min\limits_{i_L, i_U} \ i_U - i_L \\ \text{subject to:} \\ \qquad \Phi_s(i_U) - \Phi_s(i_L) \ge \alpha \\ \qquad i_L \text{ and } i_U \text{ are positive integers.} \end{cases} \tag{19}$$

Below are the details about the procedure to compute the prediction interval of active cases numbers based on the optimization model (19). For simplicity, we take the computation of prediction interval with confidence $\alpha = 0.95$ on April 15 as an example.

Step 1 Set April 11, 2020 as the initial day, and set $I_0 = 1138$ in Eq. (18) which is the number of active cases on April 11, 2020. Then for April 15, i.e., the fourth day from the initial day, we have $t = 4$.

Step 2 Set

$$\beta_I^* = 1.1745 \times 10^{-11}, \quad \sigma_1^* = 7.7866 \times 10^{-12}, \quad \mu_I^* = 0.0785, \quad \sigma_5^* = 0.0296$$

in Eq. (18) which have already been obtained in Sect. 3 based on the numbers of active cases and closed cases in Mainland China from February 12 to April 11, 2020. Then we get

$$\Phi_4(x) = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} \cdot \frac{6.7889 - \ln x}{0.1619}\right)\right)^{-1}.$$

Step 3 Set $\alpha = 0.95$, and solve the optimization problem

$$\begin{cases} \min\limits_{x_1>0, x_2>0} \quad x_2 - x_1 \\ \text{subject to:} \\ \qquad \Phi_4(x_2) - \Phi_4(x_1) \geq \alpha \end{cases}$$

by using the function "fmincon" in Matlab toolbox. The optimal solution is $(x_1, x_2) = (617.6, 1199.6)$.

Step 4 Noting that i_L and i_U are positive integers in the optimization problem (19), we have $i_L = 617$ and $i_U = 1200$ as $\Phi_4(1200) - \Phi_4(618) < 0.95$ and $\Phi_4(1199) - \Phi_4(617) < 0.95$.

Hence, with 95% confidence, the number of active cases on April 15 will be no greater than 1200 and no less than 617.

6 Conclusion

By means of uncertain differential equations, an uncertain SEIAR model was derived to describe the spread of an epidemic. Specifically, with the COVID-19 cases data released by the National Health Commission of the People's Republic of China, the parameters in an uncertain epidemic model were estimated by following the method of moments. Furthermore, a method to predict the numbers of active cases was presented. Further research may consider the influence of vaccination to susceptible individuals, and design efficient vaccination strategies based on the uncertain SEIAR model.

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