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Hints

1.3.1 Show that any two inadjacent vertices have a common neighbor.

2.1.10 The idea of diameter, from the preceding exercise, is useful. Say G has diameter d. Choose a and t of distance d. Say $a, \ldots, s, t$ is a path of length d. Show that if $G - \{s, t\}$ is disconnected, then $G - \{x, y\}$ is connected for some other pair of vertices.

2.4.6 Consider a complete bipartite graph with its two parts equal or nearly equal.

2.4.8 (i) From Theorem 2.5, it suffices to show that there do not exist nonadjacent vertices x and y with $d(x) + d(y) < v$.

3.1.7 Use Exercise 1.2.4.

3.2.3 Suppose G contains r cutpoints. Use induction on r. Consider blocks containing exactly one cutpoint.

4.1.7 Write c for the vertex of degree 4 and x, y, z, t for the four of degree 1. There are unique paths cx, cy, cz, ct. Show that every vertex other than c lies on exactly one of those paths. If any of those vertices has degree > 2, prove there is another vertex of degree 1.

4.1.8 consider a vertex of degree 1, and its unique neighbour. Now work by induction on v.

4.1.11 (ii) Use induction on the number of vertices. Given a tree T, look at the tree derived by deleting vertices of degree 1 from T.

4.3.4 Consider a tree in which the weights are the negative of those given.

6.1.8 Select a vertex x of degree 1. If yz is any edge of the tree, define the *distance* from x to yz to be the smaller of $D(x, y)$ and $D(x, z)$. Prove that every one-factor of T must contain preciselt the edges of even distance from x. If these form a one-factor, T has one; otherwise it has none. (There are other proofs.)

6.1.9 Proceed by induction on the number of vertices. Use Exercise 2.1.8.

6.3.3 Assuming G has 1 or 2 bridges, it is useful to notice that the proof of Theorem 6.10 works just as well if there were 2 edges joining the vertices x and y instead of just one. Proceed by induction on the number of vertices of G.

6.4.4 Generalize Exercise 6.4.3.

7.3.4 Use Theorem 7.7.

7.4.6 Suppose G is a graph with km edges, $k \geq \chi'(G)$. Write $\mathcal{C}$ for the set edge-colorings of G in k colors. If $\pi \in \mathcal{C}$, define $n(\pi) = \sum |e_i - m|$, where e_i is the number of edges receiving color c_i under,π, and the sum is over all colors. Then define $n_0 = \min\{n(\pi) : \pi \in \mathcal{C}\}$. Assume $n_0 > 0$ and derive a contradiction. Then a coloring achieving n_0 has the required property.

7.5.1 Verify this exhaustively. But: (i) to prove that whenever an edge is deleted the result can be 3-edge-colored, notice that there are only 3 different sorts of edge (chord, outside edge with both endpoints degree 3, outside edge with one of degree 2) (in fact, the first two are equivalent, but proving this is just as hard as checking one more case); to prove the graph requires 4 colors, notice that there are only three different ways to 3-color the top 5-cycle, and none can be completed.

8.2.2 Use Theorems 8.5 and 8.6.

9.1.9 Form a graph whose vertices are the rows $M_1, M_2, \ldots, M_s$ of M If $i < j$, then allocate a color to the edge $M_i M_j$ corresponding to the ordered pair (m_{ij}, m_{ji}).

10.1.3 Follow the proof of Theorem 2.1.

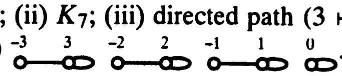
12.4.4 The maximum flow cannot exceed 14 because of the cut $[s, abcdt]$.

13.1.7 Use a function f that oscillates finitely.

Answers and Solutions

Exercises 1.1

1.1.1 (i) S; (ii) RS; (iii) A; (iv) RST.

1.1.3 Answer (iii) is a digraph, because the relation is not symmetric; the others are graphs. (i) K_7- edge 23; (ii) K_7; (iii) directed path $(3 \mapsto 2 \mapsto 1 \mapsto 0 \mapsto -1 \mapsto -2 \mapsto -3)$; (iv) 

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1.1.5 (i) S; (ii) R; (iii) S; (iv) AS.

Exercises 1.2

1.2.1 $G : I = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{matrix}\ ;\ A = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}\ .$

$H : I = \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}\ ;\ A = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}\ .$

1.2.4 Say V_1, V_2 are non-empty disjoint sets of vertices of G such that there is no edge joining any vertex of V_1 to any vertex of V_2. Concider vertices x, y of G. If one is in V_1 and the other is in V_2, then they are adjacent in $\overline{G}$. If both are in the same set, say V_1, then select any vertex z in V_2; xz and zy are edges in $\overline{G}$.

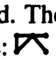
1.2.7 Say the two parts of G contain p and q vertices respectively. Then G has at most pq edges (it will have fewer unless G is complete bipartite). Moreover, $p+q=v$. Say $p=\frac{v}{2}+\pi, q=\frac{v}{2}-\pi$. Then $pq=(\frac{v}{2}+\pi)(\frac{v}{2}-\pi)$ $=\frac{v^2}{4}-\pi^2\leq\frac{v^2}{4}$.

1.2.10 $2n$; $A = \begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{matrix}$

$$I = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{matrix}.$$

Exercises 1.3

1.3.1 Suppose x and y are not adjacent. Then each of them is adjacent to at least $\frac{v-1}{2}$ of the remaining $v-2$ vertices. So they have a common neighbor, say z, and xzy is a walk in G.

1.3.3 $\{3,2,2,2,1\}$ is valid iff $\{1,1,1,1\}$ is valid. The latter corresponds to two disjoint edges, so it is valid. Two examples:  

1.3.6 (i) no (you get $\{2,2,0,0\}$); (ii) yes; (iii) no (sum is odd); (iv) yes. If d and v are natural numbers, not both odd, with $v > d$, then there there is a regular graph of degree d with exactly v vertices.

1.3.10 Consider a graph with vertices $x_1, x_2, \ldots, x_v$, where the subscripts are integers modulo v. If d is even, say $d = 2n$, the edges are
$x_ix_j : 1 \leq i \leq v, i+1 \leq j \leq i+n$.
This will yield the required graph provided $2n < v$. If d is odd, $d = 2n+1$, then v must be even. Use the same construction and add an edge $x_ix_{\frac{v}{2}+i}$ for each i.

1.3.11 (i) yes; yes. (ii) Here is one construction. Assign a vertex to each member of the sequence. Arbitrarily pair up the vertices corresponding to odd integers and join the pairs. Then, to a vertex corresponding to integer d, assign $\lfloor d/2 \rfloor$ loops.

Exercises 2.1

2.1.1 (i) $sacbt, sacbdt, sact, sbct, sbdt, sbt, scbt, scbdt, sct$; 2.

2.1.2 (i) $sa:1, sb:1, sc:1, sd:2, st:2, ab:2, ac:1, ad:3, at:2, bc:1, bd:1, bt:1, cd:2, ct:1, dt:1$.

2.1.4 (i) Suppose i is the smallest integer such that $u_i = v_j$ for some j. Then $x, u_1, \dots, u_i, v_{j-1}, \dots, v_1, x$ is a cycle unless $i = j = 1$. (ii) Consider $u_1 = v_1 = y$.

2.1.9 Clearly $D(G)$ = the maximum distance between any two vertices in G. Say x and y attain this maximum distance — $D(x, y) = D(G)$ — and say z is a vertex that attains the eccentricity — $\varepsilon(z) = R(G)$. Clearly R is a distance between two vertices, so $R \le D$. But by definition $D(z, t) \le \varepsilon(z) = R(G)$ for every vertex t, so $D = D(x, y) \le D(x, z) + D(z, y) \le 2R$.

2.1.10 The result is clearly true if G is complete. Suppose not. If G has diameter D ($D > 1$), choose two vertices a and t whose distance is D. Let $a, \dots, r, s, t$ be a path of length D from a to t in G. $a = r$ is possible. $r \not\sim t$.
Suppose $G* = G - \{s, t\}$ is not connected; say A is the component of $G*$ that contains a. Every vertex in $G * -A$ must be adjacent in G to s. (If not, suppose z were a vertex in $G * -A$ whose distance from s is at least 2. Then the shortest path from a to z must be of length at least $D + 1$, which is impossible.) If $G * -A$ contains any edge, say bc, then s is still connected to every vertex of $G * -A - \{b, c\}$; moreover s is adjacent to t and connected to every vertex in A (since A is connected and s is connected to a). So $G - \{b, c\}$ is connected and we could take $\{x, y\} = \{b, c\}$.
We need only consider the case where $G * -A$ consists of isolated vertices, all adjacent to s. If A has two elements, they together with r and s form an induced $K_{1,3}$. So $|A| = 1$. Say $A = \{w\}$. If $w \sim t$ take $\{x, y\} = \{w, t\}$, and if $b \not\sim t$ then r, s, t, w form an induced $K_{1,3}$.

Exercises 2.2

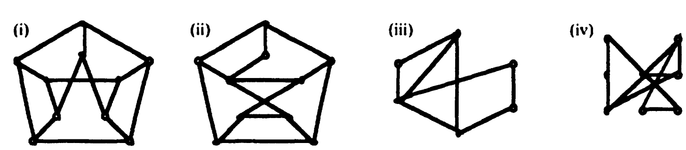
2.2.2 (i) $seft$ (length 9); (ii) $sebt$ (length 11).

Exercises 2.3

2.3.1 (i) No Euler walk, as there are 4 odd vertices; 2 edges are needed. (ii) There is a closed Euler walk. (iii) There are two odd vertices, so there is an Euler walk, but not a closed one. Two edges are needed.

Exercises 2.4

2.4.2



2.4.8 (i) From Theorem 2.5, it suffices to show that there do not exist nonadjacent vertices x and y with $d(x) + d(y) < v$. So, of the $2v - 3$ pairs xz and yz, with $z \in V(G)$, at most $v - 1$ are edges. So G has at most $\binom{v}{2} - (v-2) = \frac{v^2-3v+4}{2}$ edges.
(ii) If G is formed from K_{v-1} by adding one vertex and one edge connecting it to one of the original vertices, then G has $\frac{v^2-3v+4}{2}$ edges and is not Hamiltonian.

2.4.10 (i) One solution is to seat the people in the following sequences, where the labels are treated as integers mod 11: (i) $1, 2, 3, \ldots$; (ii) $1, 3, 5, \ldots$; (iii) $1, 4, 7, \ldots$; (iv) $1, 5, 9, \ldots$; (v) $1, 6, 11, \ldots$. In other words, on day i, the labels increase by i (mod 11). Over 5 days, x sits next to $x \pm 1, x \pm 2, x \pm 3, x \pm 4$ and $x \pm 5$, giving every possible neighbor once.

Exercises 2.5

2.5.1 $(v-1)!/2$.

2.5.4 (i) SE: *abcdea*, cost 115. NN: *acdeba*, cost 118.
(ii) SE: *abcdea*, cost 286. NN: *abcdea*, cost 286.

2.5.5 (i) *acdeba*, *bcdeab*, *cdeabc*, *dcbaed*, *ecdabe*, costs 118, 115, 115, 115, 121 respectively.
(ii) *abcdea*, *bcdeab*, *cdeabc*, *dabced*, *eabcde*, costs 286, 286, 286, 319, 286 respectively.

2.5.7 A directed graph model must be used. Replace each edge xy by two arcs xy and yx, with the cost of travel shown on each. In the nearest neighbor algorithm, one considers all arcs with *tail* x when choosing the continuation from vertex x.

Exercises 3.1

3.1.2 If it did, deleting the bridge would yield components with exactly one odd vertex.

3.1.3 (i)

$$\begin{array}{lcl@{\qquad}lcl}
[a,bcde] & = & \{ac,ad\} & [ae,bcd] & = & \{ac,ad,be,de\} \\
[ab,cde] & = & \{ac,ad,be\} & [abe,cd] & = & \{ac,ad,de\} \\
[ac,bde] & = & \{ad,cd\} & [ace,bd] & = & \{ad,be,cd,de\} \\
[abc,de] & = & \{ad,be,cd\} & [abce,d] & = & \{ad,cd,de\} \\
[ad,bce] & = & \{ac,de\} & [ade,bc] & = & \{ac,be,cd\} \\
[abd,ce] & = & \{ac,be,cd,de\} & [abde,c] & = & \{ac,cd\} \\
[acd,be] & = & \{de\} & [acde,b] & = & \{be\} \\
[abcd,e] & = & \{be,de\} & & &
\end{array}$$

(ii)

$$\begin{array}{lcl@{\qquad}lcl}
[a,bcd] & = & \{ab\} & [ad,bc] & = & \{ab,bd,cd\} \\
[ab,cd] & = & \{bc,bd\} & [abd,c] & = & \{bc,cd\} \\
[ac,bd] & = & \{ab,bc,cd\} & [acd,b] & = & \{ab,bc,bd\} \\
[abc,d] & = & \{bd,cd\} & & &
\end{array}$$

Exercises 3.2

3.2.1 Suppose G is a connected graph with at least two edges.
(i) G is connected and is not K_2, so each edge is adjacent to some other edge. So "any two adjacent edges lie on a cycle" implies that each edge lies on a cycle. So each point lies on a cycle, and there are no cutpoints.
(ii) Suppose xy and yz are adjacent edges that do not lie on any common cycle. There can be no path from x to z that does not contain y (if there were, that path plus xy and yz would be a cycle containing the two edges). So y is a cutpoint.

3.2.3 Suppose G contains r cutpoints. We proceed by induction on r. The case $r = 0$ is trivially true; the equation becomes $-1 = -1$. Assume the result is true for graphs with r or fewer cutpoints, $r \geq 0$, and suppose G has $r+1$ cutpoints. We define an *endblock* in G to be a block containing exactly one cutpoint y. Clearly G contains an endblock. Select an endblock E of G, and form a graph H by deleting from G all vertices and edges of E except for the unique cutpoint. Then $b(H) = b(G) - 1$ blocks. For each vertex x of H, $b_H(x) = b_G(x)$, except $b_H(y) = b_G(y) - 1$. The $|V(E)| - 1$ deleted vertices each belonged to 1 block of G. By induction, $b(H) - 1 = \sum_{x \in V(H)}[b_H(x) - 1] = \sum_{x \in V(H), x \neq y}[b_H(x) - 1] + b_H(y) - 1 = \sum_{x \in V(H), x \neq y}[b_G(x) - 1] + b_G(y)$. So $b(G) - 1 = \sum_{x \in V(H)}[b(x) - 1] = \sum_{x \in V(G)}[b(x) - 1]$ (the vertices of G not in H all contribute 0 to the sum, because they were all in one block of G).

Exercises 3.3

3.3.1 $\kappa, \kappa', \delta =$ (i) 1,1,1 (ii) 1,2,2 (iii) 1,1,2 : .

3.3.2 Each graph contains a spanning cycle, so each has $\kappa \geq 2$. The third has $\delta = 2$, so by Theorem 3.5 $\kappa' = 2$. In the first, removing of any vertex leaves a Hamiltonian graph, so at least two more must be deleted to disconnect, and $\kappa = 3$, whence $\kappa' = 3$. For the second graph, the preceding argument shows that the only candidates for two vertices whose removal would disconnect it are the top two in the diagram, but they do not work, so $\kappa = \kappa' = 3$. The answers are (i) 3,3, (ii) 3,3, (iii) 2,2.

3.3.5 Suppose G has $\delta(G) \geq \frac{1}{2}v(G)$ but $\kappa'(G) < \delta(G)$. Select a set S of $\kappa'(G)$ edges whose removal disconnects G; say $G - S$ consists of disjoint parts with vertex-sets X and Y. Every vertex of X has degree at least δ, and there are fewer than δ edges of G with exactly one endpoint in X (only the members of S fit this description), so there is at least one vertex in X with all its neighbors in X. So $|X| > \delta$; similarly $|Y| > \delta$; so $v(G) = |X| + |Y| > 2 \cdot \delta$, a contradiction.
An example with $v = 6, \delta = 2, \kappa' = 1$ consists of two disjoint triangles plus an edge joining a vertex of one to a vertex of the other.

Exercises 4.1

4.1.3 Suppose G is a finite acyclic graph with v vertices. If G is connected it is a tree, so it has $v-1$ edges by Theorem treesize. Now assume G has $v-1$ edges. Suppose G consists of c components $G_1, G_2, \dots, G_c$, where G_i has v_i vertices; $\sum v_i = v$. Each G_i is a tree, so it has $v_i - 1$ edges, and G has $\sum (v_i - 1) = v - c$. So $c = 1$ and G is connected.

4.1.5 If G contains edges xy and yz then G^2 contains triangle xyz. so G^2 a tree $\Rightarrow G$ consists of disjoint K_1's and K_2's $\Rightarrow G^2$ consists of disjoint K_1's and K_2's. The only trees are K_1 and K_2.

4.1.6 One example: vertices are integers, $x \sim x+1$.

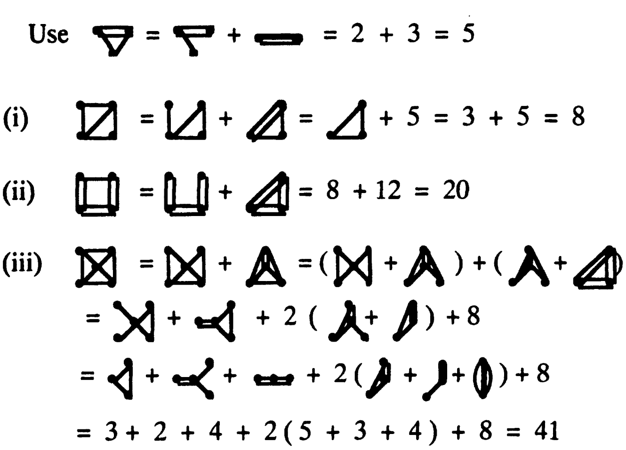
4.1.9 Suppose x has degree k. The longest path in T contains at most two edges incident with x, so there are $k-2$ edges known not to be on the pat most $(v-1)-(k-2)$ edges are available.

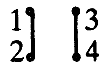
4.1.12 (ii) Suppose G is a connected self-centered graph with a cutpoint x. S... a vertex y such that $D(x, y) = \varepsilon(x)$. Let P be a shortest xy-path. Then P lies completely within some component of $G - x$. Select z, a vertex in some other component of $G - x$. Clearly $\varepsilon(z) \geq D(z, y) > D(x, y) = \varepsilon(x)$, contradicting the centrality of x.

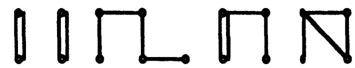
Exercises 4.2

4.2.3 The "only if" is obvious. When $v \geq 4$ there are many constructions. One example: take the vertices as $1, 2, \dots, v \pmod v$). One tree is the path $1, 2, \dots, v$. If v is even, take as the second tree the path $1, 3, \dots, v, 2, \dots, v-1$. If v is odd, take the path $1, 3, \dots, v-1, 2, 4, \dots, v$. Another example: select four different vertices x, y, z, w. One tree consists of all edges from x to another vertex other than xy, plus zy. The other consists of all edges from y to another vertex other than yz, plus wz.

4.2.4



4.2.7 If G is 

one solution is 

4.2.9 16; 125.

4.2.12 It is clearly necessary that H have no cycles, and if $H = G$ the result is immediate. So suppose H is acyclic and $H < G$. Say H has disjoint components $H_1, H_2, \ldots, H_n$. Since G is connected, there is in each H_i some vertex x_i that is adjacent to some vertex, y_i say, that is in G but not in H. Write $S = V(G)\backslash V(H)$, and select a spanning tree T in $\langle S\rangle$. Then

$$T \cup H \cup \{x_i y_i : 1 \le i \le n\}$$

is a spanning tree in G.

rcises 4.3

1 There are several solutions, but the minimum weight is (i) 54, (ii) 38, (iii) 33.

Exercises 5.1

5.1.1 (i)

+	0	1	2
0	0	1	2
1	1	0	2
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

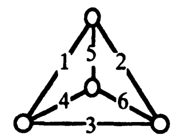
5.1.3 There is exactly one of dimension 0 ($\emptyset$), one of dimension 4 (V), and none of dimension 5. For dimension 1, the subspaces are $0, x$ where $x \neq 0$, so there are $|V| - 1 = 15$ of them. For dimension 2, any ordered pair x, y of distinct nonzero elements determine the subspace $0, x, y, x + y$. Each of these ordered bases arises 6 times if all ordered pairs are listed, so there are $15 \cdot 14/6 = 35$. For dimension 3, there are $15 \cdot 14 \cdot 12$ ordered bases. Each subspace has 8 elements, so by (5.1) it has 764 ordered bases. So the number of subspaces is $15{\cdot}14{\cdot}12/(7{\cdot}6{\cdot}4) = 15$. (Those who know a little more linear algebra will see from perpendicularity that the number of 3-dimensional subspaces must equal the number of 1-dimensional subspaces.)

Exercises 5.2

5.2.2 They form a basis if and only if n is even. Write $S = \{x_1, x_2, \ldots, x_n\}$, and $S_i = S\backslash\{x_i\}$. $\sum S_i = (n-1)S$, where n is reduced mod 2. If n is odd, the sum is zero, and the S_i are not independent. If n is even, $\sum S_i = S$, and $\sum_{i \neq j} S_i = S_j + S_j + \sum_{i \neq j} S_i = S_j + S = \{x_j\}$, so $\langle\{S_i\}\rangle$ contains all the singletons, so it contains all of S. Since there are n elements, $\{S_i\}$ is a basis.

Exercises 5.3

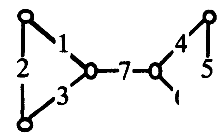
5.3.1 The cycles of K_4 are 123, 145, 256, 364, 1264, 1563, 2345. The union of any two of these is another of them. So the cycle space has $8 = 2^3$ elements (don't forget $\emptyset$), so it has dimension 3.



Exercises 5.4

5.4.1 The cycle of length 4 belongs to both. A necessary (not sufficient!) property is that the graph must have a cycle of even length.

5.4.2 (i) (a) Cycle subspace {$\emptyset$, 123, 456, 123456}, cutset subspace {$\emptyset$, 13, 23, 12, 4, 134, 234, 124, 56, 1356, 2356, 1256, 456, 13456, 13456, 23456, 12456, 57, 1357, 2357, 1257, 457, 13457, 23457, 12457, 67, 1367, 2367, 1267, 467, 13467, 23467, 12467}.



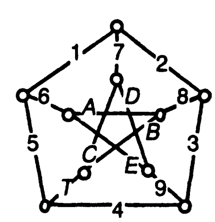
(b) Tree, so cycle subspace = $\emptyset$. Cutset subspace contains all 2^5 subsets of the edges.

(ii) (a) Cycle subspace has $4 = 2^2$ elements, dimension 2. Cutset subspace has $32 = 2^5$ elements, dimension 5. $2 + 5 = 7$. (b) Cycle subspace has $1 = 2^0$ elements, dimension 0. Cutset subspace has $32 = 2^5$ elements, dimension 5. $0 + 5 = 5$.

Exercises 5.5

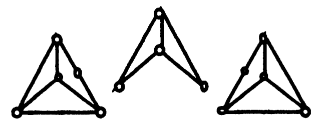
5.5.1 Choose i such that $2 \le i \le k$ and let L_i be the fundamental cycle corresponding to the edge a_i. Now a_1 is the only edge of T in C and a_i is the only edge of $\overline{T}$ in L_i. So $\{a_i\} \subseteq C \cap L_i \subseteq \{a_1, a_i\}$. By Lemma 5.3, $|L \cap C_i|$ is even, so $L \cap C_i = \{a_1, a_i\}$ whence $a_1 \in L_i$. Now let a_{k+j}, $j \ge 1$, be an edge of $\overline{T}$, and L_{k+j} the corresponding cycle. Since L_{k+j} contains no other edge of $\overline{T}$, $\emptyset \subseteq C \cap L_{k+j} \subseteq \{a_1\}$. Again by Lemma 5.3, $|L \cap C_{k+j}|$ is even, so $L \cap C_{k+j} = \emptyset$, so $a + 1 \notin C_{k+j}$.

5.5.4 (i) $1, 2, 3, 4, 5, 6, 7, 8, 9, T$.
(ii) (12345), (2347TA), (2379B), (1268C), (348TD), (12369E)
(iii) (15CE), (25$ABCE$), (35$ABDE$), (45AD), (6CE), (7AB), (8CD), (9BE), (TAD)
(iv) Cycles of length 8.



Exercises 6.1

6.1.2 Suppose N has a one-factor. One edge from the center vertex must be chosen; say it is the vertical one. Then the remaining edges of the factor must form a one-factor in the following graph, which has odd components:



6.1.5 (i) $ab\ cd\ ef\ gh$; $ae\ bc\ dh\ fg$.
(ii) $ae\ bc\ dh\ fg$; $ae\ bg\ dh\ fc$; $af\ be\ ch\ dg$.

6.1.9 Suppose G has $2n$ vertices. We proceed by induction on n. The result is true for $n = 2$ (see Exercise 6.1.1). Say it is true for $n \leq N$. Suppose $v(G) = 2N + 2$. By Exercise 2.1.10, G has an edge xy such that $G - \{x, y\}$ is connected. Now $G - \{x, y\}$ contains no induced $K_{1,3}$, and has $2N$ vertices. So by the induction hypothesis it has a one-factor. Append xy to that factor to construct a one-factor in G.

Exercises 6.2

Use a one-factorization of $K_{n,n}$. An example for $n = 4$ is
$1a\ 2b\ 3c\ 4d$, $1b\ 2a\ 3d\ 4c$, $1c\ 2d\ 3a\ 4b$, $1d\ 2c\ 3b\ 4a$.

6.2.2 (i) It is required to find edge-disjoint factors of K_v, each of which consists of $v/3$ triangles.

Exercises 6.3

6.3.1 No. For example, consider $3K_3 \cup 3K_5$.

6.3.3 If G has no bridge, Theorem 6.10 gives the result. for the cases where G has 1 or 2 bridges, it is useful to notice that the proof of Theorem 6.10 works just as well if there were 2 edges joining the vertices x and y instead of just one. We proceed by induction on the number of vertices of G. The result is trivial for 4 vertices.

If G has 1 bridge, xy, write G_x and G_y for the components of $G - xy$, with $x \in G_x$. Say the vertices adjacent to x in $G - x$ are x_1 and x_2. The (multi)graph defined by adding edge x_1x_2 to $G_x - x$ is cubic has no bridge, so it has a 1-factor not containing the new edge. (Simply insist that it contains one of the other edges incident with x_1.) So does the graph similarly derived from G_y. Add xy to the union of these factors.

If G has 2 bridges, they cannot have a common endpoint (if they did, then the third edge through that vertex would also be a bridge.) Say the bridges are xy and zt, and say the three components of $G - xy - zt$ are G_x (containing x), G_y (containing y and z), and G_t (containing t). Then G_y has an even number of vertices, while the others are odd. We can construct a one-factor containing yz in the bridgeless (multi)graph $G_y + yz$, and a one-factor including the bridge xt in $G_x \cup G_t + xt$. Their union is the required factor.

Exercises 6.4

6.4.2 There is no example for $s = 1$. For $s = 2$, $K_3 \cup P_2$ os a $1-(1,1,2)$ graph.

6.4.3 The degrees are clearly correct. But the new vertex is a cutpoint, so G is not Hamiltonian.

Exercises 7.1

7.1.1 3. ($\chi > 2$, because there is an odd cycle. 3 is easily realized.)

7.1.3 Write χ for $\chi(G)$, β for $\beta(G)$.
(i) Select a χ-coloring of G. Write V_i for the color classes. Each V_i is an endependent set, so $|V_i| \leq \beta$, so $v = \sum |V_i| \leq \chi \cdot \beta$.
(ii)Select a maximal independent set S; $|S| = \beta$. G can be colored in $\chi(G-S)+1$ colors (just color all points of S in a new color). $G - S$ has v vertices, so obviously $\chi(G-S) \leq v-\beta$. so $\chi \leq \chi(G-S)+1 \leq v-\beta$

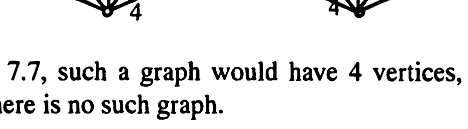
7.1.6 $x_1, x_6, x_2, x_3, x_4, x_5$ works.

7.1.8 Select one edge in the cycle, say xy. By Theorem 7.1, $\chi(G - xy) = 2$. Select a 2-coloring of $G - xy$ and apply a third color to xy.

7.1.10 Color $G - v$ in n colors. There must be a color not on any vertex adjacent to x in G. Apply that color to x.

Exercises 7.3

7.3.2 (i) Only one has a vertex of degree 2.
(ii) Neither graph has any coloring in 0, 1, 2 or 3 colors (each contains a K_4), so each has polynomial divisible by $x(x-1)(x-2)(x-3)$. For 4 colors there are 48 colorings: if colors 1, 2, 3, 4 are applied to the upper triangle, then the other colors are determined as shown. In the first graph, t can be 2 or 3, and in the second graph, (y, z) can be $(2, 3)$ or $(3, 2)$. This gives 2 colorings each, and $\{1, 2, 3, 4\}$ can be permuted in 24 ways. So each has a polynomial of the form $p(x) = x^6 - 11x^5 + \ldots = x(x-1)(x-2)(x-3)(x^2+ax+b) = x^6 + (a-6)x^5 + \ldots$. Comparing coefficients of x^5, $a - 6 = -11$, $a = -5$. Then $p(4) = 48$ reduces to $(4^2 + 4 \cdot 5 + b) = 2$, or $b = 6$. So the polynomial is $x(x-1)(x-2)(x-3)(x^2-5x+6) = x(x-1)(x-2)^2(x-3)^2$, the same for both graphs.



7.3.4 From Theorem 7.7, such a graph would have 4 vertices, 4 edges and 2 components. There is no such graph.

Exercises 7.4

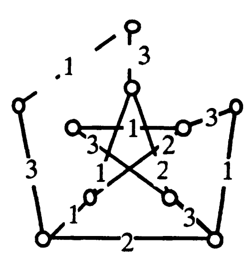
7.4.2 Any 8-edge graph on 5 vertices has $\Delta = 4$ (sum of degrees = 16). There are two such graphs, the complements of $2K_2$ and P_3. For the former, take a one-factorization of K_6, delete the edges of one factor and then delete one vertex; the remaining (partial) factors are the color classes in a 4-edge-coloring. In the latter, consider the K_5 on vertices 1, 2, 3, 4, 5 with edges 15 and 25 deleted. Suitable color classes are $\{12, 34\}$, $\{13, 24\}$, $\{14, 35\}$, $\{23, 45\}$. So the graphs both have edge-chromatic number 4, and both are class 1.

7.4.3 First, observe that any 7-edge graph on 5 vertices can be edge-colored in 4 colors, because it can be embedded in an 8-edge graph on 5 vertices (and use the preceding exercise). Now if a 7-edge graph can be edge-colored in 3 colors, one color would appear on 3 edges. But you can't have 3 disjoint edges on only 5 vertices.

7.4.6 Suppose G is a graph with km edges, $k \geq \chi'(G)$. Write $\mathcal{C}$ for the set of all edge-colorings of G in k colors. If $\pi \in \mathcal{C}$, define $n(\pi) = \sum |e_i - m|$, where e_i is the number of edges receiving color c_i under,π, and the sum is over all colors. Then define $n_0 = \min\{n(\pi) : \pi \in \mathcal{C}\}$. We prove that $n_0 = 0$. Then a coloring achieving n_0 has the required property.

Suppose $n_0 > 0$. Let π_0 be a coloring with $n(\pi_0) = n_0 > 0$. Since G has km edges, there exist color classes M_1 and M_2 under π such that $e_1 = |M_1| < m$ and $e_2 = |M_2| > m$. Say the other color classes have sizes c_3, $c_4, \ldots, c_k$. Now $M_1 \cup M_2$ is a union of paths and cycles. $e_2 > e_1 \Rightarrow$ the union includes at least one path P with its first and last edges from M_2. Exchange the colors of edges in P. The resulting edge-coloring π' has one more edge in color C_1 and one fewer in color c_2, so its color classes are of sizes $c_1 - 1, c_2 - 1, c_3, \ldots, c_k$, and $n(\pi') < n(\pi)$, a contradiction.

7.4.9 (i) By Exercise 6.1.4, $\chi'(P) > 3$, so by Theorem 7.11 $\chi'(P) = 4$.
(ii) The Figure shows a 3-edge-coloring of P − edge, so $\chi' = 3$.
(iii) delete the two broken lines from the Figure. $\chi' = 3$.

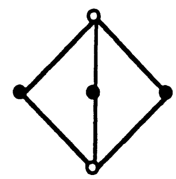


Exercises 7.5

7.5.3 Suppose G has cutpoint x and is edge critical with edge-chromatic number n. Say $G - x$ consists of two subgraphs G_1 and G_2 with common vertex x. Select vertices y in G_1 and z in G_2 adjacent to x. Choose edge-colorings π_1 of $G - xy$ and π_2 of $G - xz$ in the $n - 1$ colors $c_1, c_2, \ldots, c_{n-1}$ (possible by criticality). Permute the names of the colors in π_2 so that the π_2-colors of edges joining x to vertices of G_2 are different from the π_1-colors of edges joining x to vertices of G_1 (this must be possible: G is class 2, so the degree of x is less than n). Color the edges of G_1 using π_1 and the edges of G_2 using π_2. This is an $(n-1)$-edge-coloring — contradiction.

Exercises 8.1

8.1.3 First, convince yourself that the drawing shown of $K_{2,3}$ is quite general. Now $K_{3,3}$ can be constructed from $K_{2,3}$ by adding one vertex adjacent to the black edges. Whichever face it is placed inside, one crossing can be achieved and is unavoidable.



8.1.5 To see that P is not planar, delete the two"horizontal" edges from the resentation in figure 2.3. When the vertices of degree 2 in this subg are elided, the result is $K_{3,3}$. The crossing number is 2 (this can be sh exhaustively, starting from a representation of $K_{3,3}$ with 1 crossing).

Exercises 8.2

8.2.4 From Theorem 1.1, $2e = \sum v \geq 6v$, so $e \geq 3v$. By Theorem 8.6, G is not planar. The result follows.

Exercises 8.3

8.3.2 Suppose there are connected planar graphs that cannot be colored in six colors, and let G one with the minimum number of vertices. Let x be a vertex of G of degree less than 6. $G - x$ is 6-colorable; choose a 6-coloring ξ of $G - x$. There will be some color, say c, that is not represented among the vertices adjacent to x in G. Define $\eta(x) = c$, and $\eta(y) = \xi(y)$ if $y \in V(G - x)$. Then η is a 5-coloring of G — contradiction.

Exercises 9.1

9.1.1 (i) Clearly $R(P_3, K_3) \leq R(K_3, K_3) = 6$.
(ii) G contains no $P_3 \Leftrightarrow G$ contains no vertex of degree 2. So the components of G are disjoint vertices (degree 0) and edges (degree 1).
(iii) If G contains an isolated vertex and 4 or more components then it has 3 or more components, so $\overline{G}$ has a triangle.
(iii) suppose K_5 is colored so as to contain no red P_3 and no blue K_3. Let G be the subgraph of red edges. By (ii), (iii) $\overline{G}$ contains a K_3 unless $v \leq 4$. So $R(P_3, K_3) \leq 5$. But The K_4 with edges ab and cd red and the others blue is suitable. So $R(P_3, K_3) = 5$.

9.1.5 Say K_v contains no red or blue K_4. Select a vertex x. R_x (B_x) is the set of vertices joined to x by red (blue) edges. Then $\langle R - x\rangle$ can contain no red K_3 or blue K_4 and $|R_x| < R(3,4) = 9$. Similarly $|B_x| < 9$. So $|V(x)| \le 1 + (9-1) + (9-1) = 17$, and $R(4,4) \le 18$.

9.1.7 Suppose the edges of K_{m+n} are colored in red and blue. Any vertex x has degree $m+n-1$, so if there are less than m red edges incident with x, there must be at least n blue edges. So $R(K_{1,m}, K_{1,n}) \le m+n$.
If m or n is odd, then there exists a regular graph G of degree $m-1$ on $m+n-1$ vertices (see Exercise 1.3.10. Its complement $\overline{G}$ is regular of degree $n-1$. Color the edges of G red and those of $\overline{G}$ blue. This painting avoids any red $K_{1,m}$ and any blue $K_{1,n}$. So m or n odd $\rightarrow$ $R(K_{1,m}, K_{1,n}) = m+n$. In any painting of K_{m+n-1} that avoids both red $(K_{1,m}$ and blue $K_{1,n}$, no vertex can have more than $m-1$ red and $n-1$ blue incident edges, so each vertex has exactly $m-1$ red and $n-1$ blue, so the red chromatic subgraph is regular of degree $m-1$. This is impossible if m and n are both even (degree and order can't both be odd — Corollary 1.1.1). So m and n even $\rightarrow$ $R(K_{1,m}, K_{1,n}) < m+n$. But a painting of K_{m+n-2} is easy to find — $n-1$ is odd, so we can do it with no red $K_m - 1$ or blue K_{n-1}, let alone K_n. So m and n even $\rightarrow$ $R(K_{1,m}, K_{1,n}) = m+n-1$.

Exercises 9.2

9.2.2 (i) Suppose n is odd. Suppose the edges of K_{2n} are colored red and blue, and vertex x is incident with r red and b blue edges. If $r \ge n$, x will be the center of at least one red $(K_{1,n}$, and if $r < n$ then $b \ge n$, and x is the center of at least one blue $(K_{1,n}$. So each vertex is the center of a monochromatic star, and $N_{2.2n}(K_{1,n}) \ge 2n-1$. But if we select a regular graph of degree n on $2n$ vertices (possible by Exercise 1.3.10), and color all its edges red and insert blue edges between all inadjacent pairs, the result has exactly $2n-1$ monochromatic (red) n-stars.
(ii) Suppose n is even. Take a K_n with vertices $x_1, x_2, \dots, x_n$ and a K_{n_1} with vertices $y_1, y_2, \dots, y_{n-1}$ disjoint from it. Color the following edges red: all the edges of the K_n except x_1x_2 $x_3x_4, \dots, x_{n-1}x_n$, all the edges of the K_{n_1} and the edges x_1y_1 $x_2y_2, \dots, x_{n-1}y_{n-1}$. The other edges of K_{2n-1}r are colored blue. Every vertex of this graph has red and blue degree $n-1$ except for x_n, which has n red and $n-2$ blue edges. So there is exaclty one monochromatic K_1, n, namely $x_n - x_{n-1}y_1y_2 \dots y_{n-1}$.

Exercises 9.4

9.4.1 If a graph is to contain no red K_2, it has no red edges, so it is a blue K_v. There is no blue K_q iff $v < q$. So $R(2,q) = q$. Similarly $R(p,2) = p$.

9.4.3 Use Theorem 9.10 with $s = t = 3$. This gives
$$R_2(5) \ge (R_2(3) - 1)(R_2(3) - 1) + 1 = 26.$$

Exercises 10.1

10.1.1 (a) (i) $sa, st, as, at, bs, bt, tb$. (ii) $A(s) = \{a, t\}$, $B(s) = \{a, b\}$, $A(a) = \{s, t\}$, $B(a) = \{s\}$, $A(b) = \{s, t\}$, $B(b) = \{t\}$, $A(t) = \{b\}$, $B(t) = \{a, b, s\}$. (iii) sat, st. (iv) $satb$. (v) $\{st, at, bt\}$.
(b) (i) $sb, as, bc, ca, ce, dc, et, rd$. (ii) $A(s) = \{b\}$, $B(s) = \{a\}$, $A(a) = \{s\}$, $B(a) = \{c\}$, $A(b) = \{c\}$, $B(b) = \{t\}$, $A(c) = \{a, e\}$, $B(c) = \{b, d\}$, $A(d) = \{c\}$, $B(d) = \{t\}$, $A(e) = \{t\}$, $B(e) = \{c\}$, $A(t) = \{d\}$, $B(t) = \{e\}$. (iii) $sbcet$. (iv) $sbca$ (not unique). (v) $\{bc\}$.
(c) (i) $sa, sc, se, ab, ac, bd, ce, dc, dt, et$. (ii) $A(s) = \{a, c, e\}$, $B(s) = \emptyset$, $A(a) = \{b, c\}$, $B(a) = \{s\}$, $A(b) = \{d\}$, $B(b) = \{a\}$, $A(c) = \{e\}$, $B(c) = \{s, a, d\}$, $A(d) = \{c, t\}$, $B(d) = \{b\}$, $A(e) = \{t\}$, $B(e) = \{s, c\}$, $A(t) = \emptyset$, $B(t) = \{d, e\}$. (iii) $szbdt, sacet, scet, set$. (iv) No cycles. (v) $\{sc, se, ac, bd\}$.

10.1.5 (a) (i) DK_4, (ii) one component.
(b) (i) DK_7, (ii) one component.
(c) (i) DP_7 $sabdcet$, (ii) each vertex a different component.

10.1.8 No, it has loops.

Exercises 10.2

10.2.1 (i) Suppose the vertices are $x_1, x_2, \ldots, x_v$. Use the orientations $x_1 \to x_2$, $x_2 \to x_3, \ldots, x_{v-1} \to x_v, x_v \to x_1$. The other edges may be oriented in any way.

10.2.5 (i) 12223, (ii) 11233.

10.2.9 (i) $(xcb), (xcda), (xcbda)$. (ii) $(xdb), (xcdb), (xcdba)$.

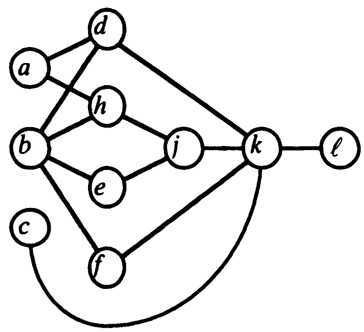
10.2.11 (i) vs = sum of the scores = sum of outdegrees. On the other hand, the sum of the outdegrees is $\binom{v}{2}$. So $vs = v(v-1)/2$ and $v = 2s + 1$.
(ii) One example: decompose K_{2s+1} into s (see Theorem 6.3), and in each cycle orient each edge in the same way around the cycle.

Exercises 10.3

10.3.2 Select an Euler walk in G. Orient each edge in the direction of the walk.

Exercises 11.1

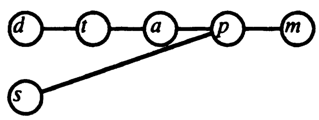
11.1.2



All arcs are directed from left to right.

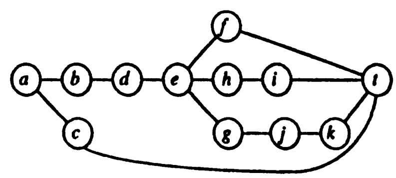
Exercises 11.2

2 (ii) 21;



All arcs are directed from left to right.

11.2.6



All arcs are directed from left to right. t is an added finish node. Critical path $abdehit$, duration 46.

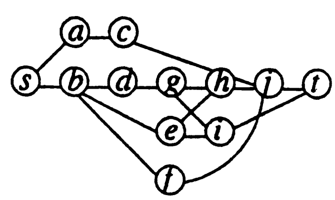
Exercises 11.3

11.3.3 Say the duration of a task in Exercise 11.2.6 was t. Then the expected time in this problem is $4t/3$ and its variance is $(t/6)^2$. The critical path is unchanged, $abdehk\ell t$, and the expected duration is $4 \cdot 46/3 = 61.33$ days. The variance is $336/6^2$, so the probability of completion within 65 days is $P(N(61.33, 3.055) \leq 65 = P(N(0, 1) \leq \frac{3.67}{3.055}$ $P(N(0, 1) \leq 1.20 = .88$.

11.3.5 Expected times: $a : 16$, $b : 13.5$, $c : 18$, $d : 8$, $e : 16$, $f : 27$, $g : 8.5$, $h : 10$, $i : 17$, $j : 9.5$. Critical path $sbfjt$, length 50. Variances $b : (\frac{5}{6})^2$, $f : 3^2$, $j : (\frac{3}{2})^2$, overall $11.9444 = 3.38^2$.

$P(N(50, 3.38) \leq 52$
$= P(N(0, 1) \leq \frac{2}{3.38}$
$= P(N(0, 1) \leq .59 = .72$.

All arcs are directed from left to right.



Exercises 12.1

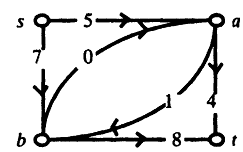
12.1.1 (i) $sadt$. (ii) $sbdt$.
(iii) (a) $\{af, df\}$, (b) $\emptyset$, (c) 11, (d) 8, (e) 5, (f) 14.

12.1.4 (i) No: imbalance at b, g. (ii) Yes.

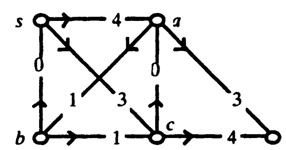
Exercises 12.2

12.2.3 (ii) 6. (iii) $sabt$. (iv) Change to $f(sa) = 7$, $f(ab) = 5$, $f(bt) = 5$, other flows unchanged. This has value 10. (v) Augment along $suxyzt$ – $f(us) = 0$, $f(xu) = 0$, $f(xy) = 4$, $f(yz) = 4$, $f(zt) = 3$. Value is 11. (vi) 11 is maximal because $[saux, bvwyzt]$ is a cut of capacity 11.

12.2.5 (i) $c[s, abt] = 5$, $c[sa, bt] = 14$, $c[sb, at] = 22$, $c[sab, t] = 12$. Minimum = 12. A flow of value 12 is shown.



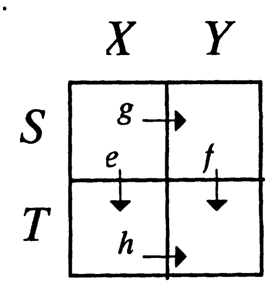
(ii) $c[s, abct] = 7$, $c[sa, bct] = 8$, $c[sb, act] = 8$, $c[sab, ct] = 7$, $c[sc, abt] = 8$, $c[sac, bt] = 8$, $c[sbc, at] = 13$, $c[sabc, t] = 8$. Minimum = 7. A flow of value 7 is shown.



12.2.7 Replace x by two vertices, x_1 and x_2. Every arc into x becomes an arc into x_1; every arc out of x becomes an arc out of x_2; and there is an arcx_1x_2 of capacity d.

Exercises 12.3

12.3.2 First, observe that both are separating cuts:
$T \cap Y = \overline{S} \cap \overline{T} = \overline{S \cup T}$; $T \cup Y = \overline{S} \cup \overline{T} = \overline{S \cap T}$,
$s \in S, X \Rightarrow s \in S \cup X, S \cap X$; $t \in T, Y \Rightarrow t \in T \cap Y, T \cap Y$.
It is easiest to draw a diagram and use single letters to represent the capacities of edges between different sets of nodes. Write:
$c[S \cap X, T \cap X] = e$, $c[S \cap Y, T \cap Y] = f$,
$c[S \cap X, S \cap Y] = g$, $c[T \cap X, T \cap Y] = h$.
Then $c[S, T] = e + f$, $c[X, Y] = g + h$, so by minimality $e + f = g + h = m$, where m is the minimal cut size. So $e + g + f + h = 2m$. Now $c[S \cup X, T \cap Y] = f + h \le m$, by minimality, and also $c[S \cap X, T \cup Y] = e + g \le m$. The only possibility is that both capacities equal m.

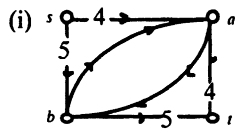


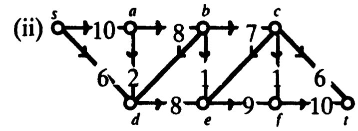
Exercises 12.4

12.4.1 There is a cut, $[sabde, cft]$, of capacity 8.

12.4.2 8 ($[s, abcdeft]$ is a cut of capacity 8).

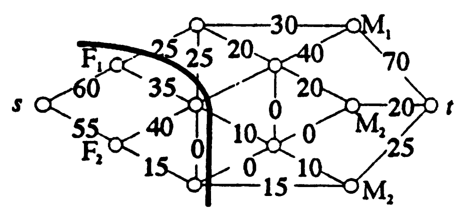
12.4.5 Max flow values are 9 and 16. Examples of flows realizing these:

(i) 

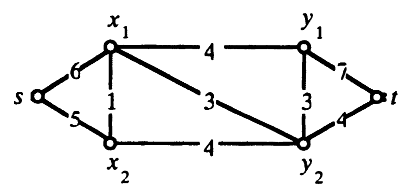
(ii) 

Exercises 12.5

12.5.2 Since there is no restriction on production or sales, add vertices s and t and put infinite capacity on all arcs sFi and Mit. Then carry out the algorithm. The maximum flow is 115; an example is shown (directions assumed to be as in the original). To see that this is maximum, observe the cut of capacity 115 shown by the heavy line.



12.5.3 Yes. A suitable flow is shown in the Figure. (Again, directions are assumed to be as in the original.)



Exercises 13.1

13.1.1 If $f = O(g)$ and $g = O(h)$ then there exist a values n_{01} and n_{02} and positive constants K_1 and K_2 such that $f(n) \le K_1 g(n)$ whenever $n \ge n_{01}$ and $g(n) \le K_2 h(n)$ whenever $n \ge n_{02}$. So, if $n \ge \max\{n_{01}, n_{02}\}$, $f(n) \le K_1 g(n) \le g(n) \le K_1 K_2 h(n)$. So $f = O(h)$ (using $n_0 = \max\{n_{01}, n_{02}\}$, $K = K_1 K_2$).

13.1.8 In testing whether n is prime, one is answering the decision problem: *is n in the set P_n, where*

$$P_n = \{x : x \le n, x \text{ is prime}\}.$$

Since $\mathcal{P}_n$ is asymptotically equal to $\sqrt{n}$, the input size of the problem is $\log n$, not n. If we write $t = \log n$ then $\sqrt{n} = e^{t/2}$, so the problem is actually exponential in the input size.

Exercises 13.3

13.3.2 (i) It is easy to see that $w_{k;ij}$ is the length of the shortest path from x_i to x_j among all paths that contain at most k edges, as required (this can be written formally as an induction). As no path can contain more than $v - 1$ edges, W_v is the matrix of shortest paths.
(ii) In the algorithm, replace line 4. by:
4. **for** $k = 1$ **to** $v - 1$ **do**
and replace line 7. by:
7. **for** $h = 1$ **to** v **do**
8. $w_{k;ij} \leftarrow \min\{w_{k-1;ij}, \min_h\{w_{k-1;ih} + w_{hj}\}\}$.
(iii) Complexity is v^4.

13.3.5 x_0 is the arbitrarily chosen starting vertex. At any stage, S is the set of vertices and T is the set of edges already selected for the tree. For vertex $y \in V\backslash S$, $W(y)$ is the minimum weight of edges joining y to S

1. $T \leftarrow \emptyset$
2. $S \leftarrow \{x_0\}$
3. **for all** $y \in V\backslash S$ **do** $W(y) \leftarrow \min_{x \in S} w(x, y)$
4. $e_y \leftarrow$ an edge xy such that $W(y) = w(x, y)$
5. **while** $S \neq V$ **do**
6.   **begin**
7.   **select** $y_0 \in V\backslash S)$
8.   **for all** $y \in V\backslash S$ **do**
9.   **if** $W(y < W(y_0)$ **then** $y_0 \leftarrow y$
10. $S \leftarrow S \cup \{y_0\}$
11. $T \leftarrow T \cup \{e_{y_0}\}$
12.   **for all** $y \in V\backslash S$ **do** $W(y) \leftarrow \min\{W(y), w(y, y_0)$
13.   **end**

This is order v^2: the main part, beginning with step 6, is of complexity v (steps 8 and 12 are both of order v, and 6 is carried out $v - 1$ times.

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