Appendix A
The Circle Bundle Point of View

The goal of this appendix is to compare the line bundle version of geometric quantisation and Berezin–Toeplitz operators with the circle bundle version of this theory. To this effect, we begin by recalling some useful facts about $\mathbb{T}$-principal bundles with connections. Then, we discuss the Hardy space and the Szegő projector of a strictly pseudoconvex domain. Finally, we explain how this enters the picture of geometric quantisation. For this appendix, we assume from the reader a basic knowledge of Lie groups and their representations.

A.1 $\mathbb{T}$-Principal Bundles and Connections

Let G be a Lie group and let X be a manifold.

Definition A.1.1. A *G-principal bundle* over X (or principal bundle over X with structure group G) is the data of a manifold P (the total space) and a smooth projection $\pi : P \to X$ together with an action of G on P such that

(1) G acts freely and transitively on P on the right: $(p, g) \in P \times G \mapsto pg \in P$,
(2) X is the quotient of P by the equivalence relation induced by this action, and π is the canonical projection,
(3) P is locally trivial in the sense that each point $x \in X$ has a neighbourhood U such that there exists a diffeomorphism

$$\varphi\colon \pi^{-1}(U) \to U \times G$$

of the form $\varphi(p) = \big(\pi(p), \psi(p)\big)$, where the map $\psi\colon \pi^{-1}(U) \to G$ is such that $\psi(pg) = \psi(p)g$ for every $p \in \pi^{-1}(U)$ and $g \in G$.

Let $P \to X$ be a principal bundle with structure group G, and let $\phi\colon G \to \mathrm{GL}(V)$ be a representation of G on some vector space V. There is a free action of G on $P \times V$ on the right:

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$$(p, v, g) \in P \times V \times G \mapsto (p, g)v := \big(pg, \phi(g^{-1})v\big) \in P \times V.$$

This action induces an equivalence relation on $P \times V$; by taking the quotient, we obtain a vector bundle $(P \times V)/G \to P/G = X$ whose fibres $(G \times V)/G$ are isomorphic to V.

Definition A.1.2. We denote by $P \times_\phi V \to X$ the vector bundle $(P \times V)/G \to X$, and we call it the vector bundle associated with the G-principal bundle $P \to X$ and the representation ϕ.

$\mathbb{T}$-Principal Bundles

Let $P \to X$ be a principal bundle with structure group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and projection π. The action of $\theta \in \mathbb{T}$ will be denoted by

$$(p, \theta) \in P \times \mathbb{T} \mapsto R_\theta(p) \in P.$$

To this action is associated the vector field ∂_θ of P defined as

$$\forall p \in P \quad \partial_\theta(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_t(p)$$

whose flow at time t is equal to R_t. The elements of $\ker(d_p\pi) = \mathrm{span}\big(\partial_\theta(p)\big)$ are called the *vertical* tangent vectors.

Definition A.1.3. A *connection* on $P \to X$ is the data of a one-form $\alpha \in \Omega^1(P)$ which is $\mathbb{T}$-invariant ($R_\theta^*\alpha = \alpha$ for every $\theta \in \mathbb{T}$) and satisfies $i_{\partial_\theta}\alpha = 1$.

A connection $\alpha \in \Omega^1(P)$ induces a splitting

$$T_pP = \ker(\alpha_p) \oplus \mathrm{span}\big(\partial_\theta(p)\big) = \ker(\alpha_p) \oplus \ker(d_p\pi).$$

The elements of the hyperplane $\ker(\alpha_p)$ of T_pP are called the *horizontal* tangent vectors. Since α is $\mathbb{T}$-invariant, the distribution $\ker\alpha$ also is, and the data of a connection is equivalent to the data of a $\mathbb{T}$-invariant subbundle E of TP such that $TP = E \oplus \ker(\mathrm{d}\pi)$. By construction, the restriction of $\mathrm{d}_p\pi$ to the horizontal subspace at p is bijective. Thus, given a vector field Y on X, there exists a unique vector field Y^{hor} on P which is horizontal and satisfies $\mathrm{d}\pi(Y^{\mathrm{hor}}) = Y$; it is called the *horizontal lift* of Y.

The connections of the trivial $\mathbb{T}$-principal bundle $X \times \mathbb{T}$ are the one-forms of the type $\beta + \mathrm{d}\theta$, where $\beta \in \Omega^1(X)$ and $\mathrm{d}\theta$ is the usual 1-form of $\mathbb{T}$.

$\mathbb{T}$-Principal Bundles and Hermitian Line Bundles

Let $L \to X$ be a Hermitian complex line bundle, and let $h(\cdot, \cdot)$ denote its Hermitian form. Let us consider the subbundle of L consisting of elements of norm 1:

$$P = \{u \in L \mid h(u,u) = 1\}.$$

One readily checks that P is a $\mathbb{T}$-principal bundle over X, with $\mathbb{T}$-action given by $R_\theta(u) = \exp(\mathrm{i}\theta)u$. Moreover, L is the vector bundle associated with P and the representation

$$\theta \in \mathbb{T} \mapsto (z \mapsto \exp(-\mathrm{i}\theta)z) \in \mathrm{GL}(\mathbb{C})$$

of $\mathbb{T}$. There is a natural isomorphism of $\mathcal{C}^\infty(X)$-modules

$$\phi\colon \mathcal{C}^\infty(X,L) \to \{f \in \mathcal{C}^\infty(P) \mid R_\theta^* f = \exp(-\mathrm{i}\theta) f\}, \quad s \mapsto f = \phi(s)$$

where, for $u \in P$, $f(u)$ is the unique complex number such that

$$s\big(\pi(u)\big) = f(u)u$$

where $\pi\colon P \to X$ is the canonical projection. Given any connection $\alpha \in \Omega^1(P)$ on P, we consider the connection ∇ on L such that the covariant derivative with respect to a vector field corresponds to the Lie derivative with respect to its horizontal lift:

$$\forall Y \in \mathcal{C}^\infty(X,TX), \forall s \in \mathcal{C}^\infty(X,L) \quad \phi(\nabla_Y s) = \mathcal{L}_{Y^{\mathrm{hor}}}\big(\phi(s)\big).$$

This map ∇ is well-defined because ϕ is an isomorphism, and it satisfies the Leibniz rule because the Lie derivative does and ϕ^{-1} is $\mathcal{C}^\infty(X)$-linear.

Exercise A.1.4. Carefully check all the above statements.

Lemma A.1.5. *The map sending α to ∇ is a bijection from the set of connections on P to the set of connections on L.*

Proof. Let us work with local trivialisations. Let $U \subset X$ be an open subset endowed with a unitary frame $s \in \mathcal{C}^\infty(U,L)$. We get a local trivialisation of P over U,

$$\varphi\colon P_{|U} \to U \times \mathbb{T}, \quad u \mapsto (\pi(u), \theta)$$

where θ is the unique element of $\mathbb{T}$ such that $s\big(\pi(u)\big) = \exp(\mathrm{i}\theta)u$. Now, let us identify $\mathcal{C}^\infty(U,L)$ with $\mathcal{C}^\infty(U)$ by sending the section fs to f, and $\mathcal{C}^\infty(P_{|U})$ with $\mathcal{C}^\infty(U \times \mathbb{T})$ via φ. Then $\phi(f) = g$ with

$$g(x,\theta) = f(x)\exp(-\mathrm{i}\theta).$$

Using these identifications, $\alpha = \beta + \mathrm{d}\theta$ for some $\beta \in \Omega^1(U)$. Therefore, given some vector field Y on U, its horizontal lift is given by $Y^{\mathrm{hor}} = Y - \beta(Y)\partial_\theta$, hence

$$(\mathcal{L}_{Y^{\mathrm{hor}}}\, g)(x,\theta) = \left(\mathrm{d}_x g(Y) - \beta(Y)\frac{\partial g}{\partial \theta}\right)(x,\theta) = (\mathcal{L}_Y\, f + \mathrm{i}\beta(Y) f)(x)\exp(\mathrm{i}\theta)$$

Consequently,

$$\nabla(fs) = (\mathrm{d}f + i\beta) \otimes s$$

so ∇ is uniquely determined by α. $\square$

A.2 The Szegő Projector of a Strictly Pseudoconvex Domain

Let Y be a complex manifold of complex dimension $n+1$. Let $D \subset Y$ be a domain (connected open subset) of Y with smooth compact boundary, defined as

$$D = \{y \in Y \mid \eta(y) < 0\}$$

with $\eta\colon Y \to \mathbb{R}$ smooth and such that $\mathrm{d}\eta(y) \neq 0$ whenever y belongs to ∂D. Let H be the complex subbundle of $T(\partial D) \otimes \mathbb{C}$ consisting of the holomorphic tangent vectors of Y which are tangent to the boundary of D; it has complex dimension n. The *Levi form* of D is the restriction to H of the quadratic form $\partial\bar{\partial}\eta$.

Definition A.2.1. We say that D is *strictly pseudoconvex* if its Levi form is positive definite at every point of ∂D.

Note that this implies that the restriction α of $-\mathrm{i}\partial\eta$ to ∂D is a contact form on ∂D. Thus we get a volume form $\mu = \alpha \wedge (\mathrm{d}\alpha)^n$ on ∂D, and we can consider the Hilbert space $L^2(\partial D)$ with respect to μ. The subspace

$$\mathcal{H}(D) = \{f \in L^2(\partial D) \mid \forall Z \in \mathcal{C}^\infty(\partial D, H)\, \mathcal{L}_{\overline{Z}} f = 0\}$$

is called the *Hardy space* of D. The *Szegő projector* of D is the orthogonal projector $\Pi\colon L^2(\partial D) \to \mathcal{H}(D)$.

A.3 Application to Geometric Quantisation

Coming back to our problem, where M is a compact Kähler manifold and $L \to M$ is a prequantum line bundle, let us introduce the $\mathbb{T}$-principal bundle $P \to M$ which consists of unit norm elements (with respect to the norm induced by h) of the line bundle L. It is such that for every integer k, we have the line bundle isomorphism $L^k \simeq P \times_{s_k} \mathbb{C}$ where $s_k\colon \mathbb{T} \to \mathrm{GL}(\mathbb{C})$ is the representation given by

$$s_k(\theta) \cdot v = \exp(-\mathrm{i}k\theta)v$$

We can embed P into $L^{-1} \simeq P \times_{s_{-1}} \mathbb{C}$ via

$$\iota\colon P \to P \times_{s_{-1}} \mathbb{C}, \quad \iota(p) = [p, 1]$$

where the square brackets stand for equivalence class. The connection on L^{-1}, that we still denote by ∇, induces a connection one-form $\alpha \in \Omega^1(P)$. Let $\mathrm{Hor}^{1,0}$ be the subbundle of $TP \otimes \mathbb{C}$ consisting of the horizontal lifts of the holomorphic vectors of $TM \otimes \mathbb{C}$. Let

$$\rho\colon L^{-1} \to \mathbb{R}, \quad u \mapsto \|u\|^2$$

and let $D = \{u \in L^{-1} \mid \rho(u) < 1\}$.

Proposition A.3.1. *D is a strictly pseudoconvex domain of L^{-1} and $\partial D = \iota(P)$. The bundle H of holomorphic vectors of L^{-1} that are tangent to $\iota(P)$ is $\iota_* \operatorname{Hor}^{1,0}$. Moreover, $\iota^* \partial \log \rho = \mathrm{i}\alpha$.*

Proof. We begin by proving the second assertion. Let us use some local coordinates. Let $U \subset M$ be an open subset such that $P_{|U} \simeq U \times \mathbb{T}$, and let us use coordinates (x, θ) on $U \times \mathbb{T}$. Then $\alpha = \beta + \mathrm{d}\theta$ for some $\beta \in \Omega^1(U)$. Let s^{-1} be the local section of $L^{-1} \to U$ defined by

$$s^{-1}(x) = [(x, 0), 1] \in (U \times \mathbb{T}) \times_{s^{-1}} \mathbb{C} \simeq L^{-1}_{|U}.$$

Then $\nabla s^{-1} = \mathrm{i}\beta \otimes s^{-1}$. We pick a function $\phi \in \mathcal{C}^\infty(U)$ such that

$$\bar{\partial}\phi + \mathrm{i}\beta^{(0,1)} = 0; \tag{A.1}$$

we know that such a function exists (taking a smaller U if necessary) thanks to the Dolbeault–Grothendieck lemma, since $\mathrm{d}\beta$ is a $(1,1)$-form. Then

$$\nabla(\exp(\phi)s^{-1}) = \exp(\phi)(\partial\phi + \bar{\partial}\phi + \mathrm{i}\beta) \otimes s^{-1} = \exp(\phi)\big(\partial\phi + \mathrm{i}\beta^{(1,0)}\big) \otimes s^{-1}$$

hence $\exp(\phi)s^{-1}$ is a holomorphic section. Let w be the complex linear coordinate of L^{-1} such that $w(\exp(\phi)s^{-1}) = 1$, and let $(z_j)_{1 \leq j \leq n}$ be a system of complex coordinates on U. In these coordinates, the maps ι and ρ read

$$\iota\colon U \times \mathbb{T} \to U \times \mathbb{C}, \quad (z_1, \ldots, z_n, \theta) \mapsto \Big(z_1, \ldots, z_n, w = \exp\big(\mathrm{i}\theta - \phi(z)\big)\Big)$$

and

$$\rho\colon U \times \mathbb{C} \to \mathbb{R}, \quad (z_1, \ldots, z_n, w) \mapsto |w|^2 \exp\big(\phi(z) + \bar{\phi}(z)\big).$$

Let $j \in [\![1, n]\!]$; the horizontal lift of ∂_{z_j} is

$$\partial_{z_j}^{\mathrm{hor}} = \partial_{z_j} - \beta(\partial_{z_j})\partial_\theta$$

We compute

$$\beta(\partial_{z_j}) = \beta^{(1,0)}(\partial_{z_j}) = -\mathrm{i}\frac{\partial\bar{\phi}}{\partial z_j},$$

the last equality coming from the fact that $\partial\bar{\phi} - \mathrm{i}\beta^{(1,0)} = 0$ because β is real-valued and satisfies (A.1). Hence

$$\partial_{z_j}^{\mathrm{hor}} = \partial_{z_j} + \mathrm{i}\frac{\partial\bar{\phi}}{\partial z_j}\partial_\theta.$$

Therefore, its pushforward by ι satisfies

$$\iota_*\big(\partial_{z_j}^{\mathrm{hor}}\big) = \mathrm{d}z_j\left(\partial_{z_j} + \mathrm{i}\frac{\partial\bar{\phi}}{\partial z_j}\partial_\theta\right)\partial_{z_j} + \mathrm{d}w\left(\partial_{z_j} + \mathrm{i}\frac{\partial\bar{\phi}}{\partial z_j}\partial_\theta\right)\partial_w,$$

which yields

$$\iota_*\big(\partial_{z_j}^{\mathrm{hor}}\big) = \partial_{z_j} + \mathrm{d}w\bigg(\partial_{z_j} + \mathrm{i}\frac{\partial\bar{\phi}}{\partial z_j}\partial_\theta\bigg)\partial_w.$$

Since $\mathrm{d}w = w(\mathrm{i}\mathrm{d}\theta - \mathrm{d}\phi)$, we finally obtain that

$$\iota_*\big(\partial_{z_j}^{\mathrm{hor}}\big) = \partial_{z_j} - \frac{\partial(\phi+\bar{\phi})}{\partial z_j}\partial_w.$$

This implies that $\iota_*\,\mathrm{Hor}^{1,0}$ is a subbundle of the bundle H of holomorphic vectors of L^{-1} which are tangent to $\iota(P)$; since both bundles have complex dimension n, this means that they are equal.

Let us now prove the last claim of the proposition. We have that

$$\partial\rho = \exp\big(\phi+\bar{\phi}\big)\Big(\overline{w}\,\mathrm{d}w + |w|^2\partial\big(\phi+\bar{\phi}\big)\Big),$$

hence

$$\partial(\log\rho) = \frac{\mathrm{d}w}{w} + \partial\big(\phi+\bar{\phi}\big).$$

Consequently,

$$\iota^*\partial(\log\rho) = \mathrm{i}\mathrm{d}\theta - \mathrm{d}\phi + \partial\big(\phi+\bar{\phi}\big) = \mathrm{i}\mathrm{d}\theta - \bar{\partial}\phi + \partial\bar{\phi}.$$

Remembering (A.1) and the conjugate equality, we finally obtain that

$$\iota^*\partial(\log\rho) = \mathrm{i}(\mathrm{d}\theta + \beta) = \mathrm{i}\alpha.$$

It remains to show that D is strictly pseudoconvex. Its Levi form is equal to the restriction of $\iota^*(\partial\bar{\partial}\log\rho)$ to $H = \iota_*\,\mathrm{Hor}^{1,0}$. But

$$\iota^*(\partial\bar{\partial}\log\rho) = -\iota^*(\bar{\partial}\partial\log\rho) = -\iota^*(\mathrm{d}\bar{\partial}\log\rho) = -\,\mathrm{d}\iota^*(\partial\log\rho) = -\mathrm{i}\mathrm{d}\alpha.$$

Since $-\mathrm{i}\mathrm{d}\alpha$ corresponds to the curvature of the connection on L over U, we have that

$$-\mathrm{i}\mathrm{d}\alpha\big(\partial_{z_j}^{\mathrm{hor}}, \partial_{\bar{z}_\ell}^{\mathrm{hor}}\big) = -\mathrm{i}\omega(\partial_{z_j}, \partial_{\bar{z}_\ell}) > 0,$$

which concludes the proof. □

As a consequence of this result, we construct the Hilbert space $L^2(P)$ by using the volume form $\mu_P = (1/(2\pi n!))\alpha\wedge(\mathrm{d}\alpha)^n$, the Hardy space

$$\mathcal{H}(P) = \{f \in L^2(P) \mid \forall Z \in \mathcal{C}^\infty(P,H), \mathcal{L}_{\overline{Z}}\, f = 0\} \subset L^2(P)$$

as in the previous section and the Szegő projector $\Pi\colon L^2(P) \to \mathcal{H}(P)$.

Since $L^k \simeq P \times_{s_k} \mathbb{C}$, we have an identification

$$\mathcal{C}^\infty(M, L^k) \to \{f \in \mathcal{C}^\infty(P) \mid R_\theta^* f = \exp(\mathrm{i}k\theta) f\}$$

which sends $s \in \mathcal{C}^\infty(M, L^k)$ to $f \in \mathcal{C}^\infty(P)$, where, for $p \in P$, $f(p)$ is the unique complex number such that $s\big(\pi(p)\big) = f(p)p$.

Lemma A.3.2. *This identification is compatible with the scalar products on $\mathcal{C}^\infty(P)$ and $\mathcal{C}^\infty(M, L^k)$ (i.e., it defines an isometry).*

Proof. Let $s, t \in \mathcal{C}^\infty(M, L^k)$ and let $f, g \in \mathcal{C}^\infty(P)$ be the corresponding functions. Observe that for $p \in P$,

$$h_k\Big(s\big(\pi(p)\big), t\big(\pi(p)\big)\Big) = f(p)\bar{g}(p)$$

since $h(p, p) = 1$. Therefore, we have that

$$\langle f, g\rangle_P = \int_P f\bar{g}\,\mu_P = \int_P \pi^*\big(h_k(s,t)\big)\,\mu_P.$$

Since $\alpha \wedge (\mathrm{d}\alpha)^n = \mathrm{d}\theta \wedge \pi^*\omega^n$, we deduce from this equality that

$$\langle f, g\rangle_P = \int_M h_k(s,t)\,\mu = \langle s, t\rangle_k,$$

which was to be proved. □

Under this identification, the covariant derivative $\nabla_X s$ corresponds to the Lie derivative $\mathcal{L}_{X^{\mathrm{hor}}} f$; hence, s is holomorphic if and only if f belongs to $\mathcal{H}(P)$, since, as we saw earlier, $H = \iota_* \mathrm{Hor}^{1,0}$. By Fourier decomposition, we have the splitting

$$L^2(P) = \bigoplus_{k\in\mathbb{Z}} \{f \in L^2(P) \mid \forall\theta \in \mathbb{T}, R_\theta^* f = \exp(\mathrm{i}k\theta) f\}.$$

To be more precise, $(R_\theta^*)_{\theta\in\mathbb{T}}$ is a family of commuting self-adjoint operators acting on $L^2(P)$, each R_θ^* has discrete spectrum $\big(\exp(\mathrm{i}k\theta)\big)_{k\in\mathbb{Z}}$, therefore they all have the same eigenspaces, and $L^2(P)$ splits into the direct sum of these eigenspaces. Now, using the above lemma, this yields a unitary isomorphism

$$L^2(P) \simeq \bigoplus_{k\in\mathbb{Z}} L^2(M, L^k).$$

Since Π commutes with every R_θ^*, $\theta \in \mathbb{T}$, we also obtain the unitary equivalence

$$\mathcal{H}(P) \simeq \bigoplus_{k\in\mathbb{Z}} H^0(M, L^k) = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k = \bigoplus_{k\geq 0} \mathcal{H}_k,$$

where the last equality comes from Proposition 4.2.1, and Π_k corresponds to the Fourier coefficient at order k of Π, that is its restriction to the space $L^2(M, L^k)$.

One can use this approach to derive another proof of Theorem 7.2.1, in a way that we quickly describe now. In their seminal article [34], Boutet de Monvel and Sjöstrand obtained a precise description of the Schwartz kernel of this projector, that we describe now. Let $\phi \in \mathcal{C}^\infty(Y \times Y)$ be such that

$$\phi(y,y) = -\mathrm{i}\eta, \quad \phi(x,y) = -\overline{\phi(y,x)}, \quad \mathcal{L}_{\overline{Z}_\ell}\phi \equiv \mathcal{L}_{Z_r}\phi \equiv 0 \bmod \mathcal{I}^\infty\big(\mathrm{diag}(Y^2)\big)$$

for every holomorphic vector field Z, where Z_ℓ (respectively Z_r) means acting on the left (respectively right) variable, and $\mathcal{I}^\infty\big(\mathrm{diag}(Y^2)\big)$ is the set of functions vanishing to infinite order along the diagonal of Y^2. It is known that such a function ϕ exists and is unique up to a function vanishing to infinite order along the diagonal of Y^2.

Define $\varphi \in \mathcal{C}^\infty(\partial D \times \partial D)$ as the restriction of ϕ to $\partial D \times \partial D$. Then $\mathrm{d}\varphi$ does not vanish on $\mathrm{diag}(\partial D \times \partial D)$, whereas $\mathrm{d}(\Im\varphi)$ vanishes on $\mathrm{diag}(\partial D \times \partial D)$ and has negative Hessian with kernel $\mathrm{diag}(T\partial D \times T\partial D)$. Thus we may assume, by modifying φ outside a neighbourhood of $\mathrm{diag}(\partial D \times \partial D)$ if necessary, that $\Im\varphi(u_\ell, u_r) < 0$ if $u_\ell \neq u_r$.

Theorem A.3.3. *([34, Theorem 1.5]) The Schwartz kernel of the Szegő projector Π satisfies*

$$\Pi(u_\ell, u_r) = \int_{\mathbb{R}^+} \exp\big(i\tau\varphi(u_\ell, u_r)\big)\; s(u_\ell, u_r, \tau)\, d\tau + f(u_\ell, u_r)$$

where $f \in \mathcal{C}^\infty(\partial D \times \partial D)$ and $s \in S^n(\partial D \times \partial D \times \mathbb{R}^+)$ is a classical symbol having the asymptotic expansion

$$s(u_\ell, u_r, \tau) \sim \sum_{j \geq 0} \tau^{n-j} s_j(u_\ell, u_r).$$

Theorem 7.2.1 can be inferred from this result, the idea being that one can deduce the asymptotics of Π_k when k goes to infinity from the description of the Schwartz kernel of Π, in a way which is similar to the deduction of the behaviour of the Fourier coefficients of a function at $\pm\infty$ from the regularity of this function. For a detailed proof using this approach, one can, for example, look at Section 3.3 in [14].

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