Appendix A
The Dispersion Relation

The dispersion relation can be derived by calculating the contour integral of Fig. A.1. Using the Cauchy formula for the function $\Pi(s)$ (which contains a cut in the positive region of the real axis and is analytic in the rest of the complex plane of s), we can obtain

$$\begin{aligned}\Pi(q^2) &= \frac{1}{2\pi i}\oint_C ds \frac{\Pi(s)}{s-q^2} \\ &= \frac{1}{2\pi i}\oint_{|s|=R} ds \frac{\Pi(s)}{s-q^2} + \frac{1}{2\pi i}\int_0^R ds \frac{\Pi(s+i\varepsilon)-\Pi(s-i\varepsilon)}{s-q^2}, \end{aligned} \tag{A.1}$$

where R is the radius of the outer circle in Fig. A.1. If R is taken to infinity, the first integral in Eq. (A.1) vanishes if $\Pi(q^2)$ decreases sufficiently fast at $|q^2| \sim R \to \infty$. If this is not the case, subtraction terms have to be considered, as will be discussed below. The second integral in Eq. (A.1) can be simplified with the help of the Schwarz reflection principle (Morse and Feshbach 1953; Arfken 1970):

$$\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon) = 2i\mathrm{Im}\Pi(s+i\varepsilon). \tag{A.2}$$

Let us examine this relation in detail. For this, we consider a complex function $f(z)$ (representing $\Pi(s)$) which is real on the real axis below a certain threshold point x_{th}, but possesses an imaginary part above x_{th}. Moreover, it is analytic and continuous on the upper half of the imaginary plane (we call this region D_1) except the region on the real axis above x_{th}. Now, we define the function $g(z)$, defined in the lower half of the imaginary plane (D_2) such that $g(z) = \overline{f(z)}$ (Here, the bar stands for complex conjugation). It can be easily shown that the real and imaginary parts of $g(z)$ satisfy the Cauchy-Riemann conditions, which means that it is an analytic function in D_2. Furthermore, $g(z)$ equals $f(z)$ on the real axis below x_{th}, because $f(z)$ is real there. Now, from the theory of analytic continuation of analytic functions, one can proof the following theorem (Morse and Feshbach 1953):

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3,

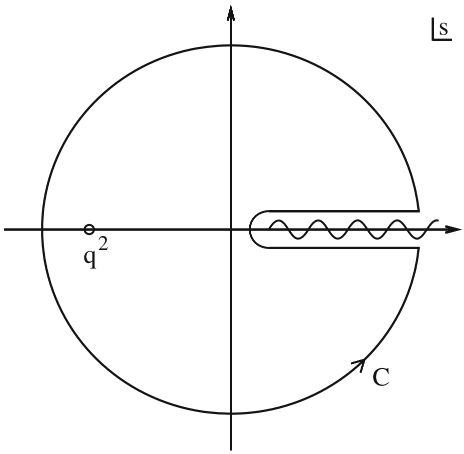

Fig. A.1 The contour integral C on the complex plane of the variable s, used for deriving the dispersion relation of Eq. (3.2). The *wavy line* denotes the non-analytic cut of $\Pi(s)$ on the positive side of the real axis. Note that here q^2 takes a negative value

> If f is analytic in D_1 and g in D_2, if f equals g along their common boundary A, and if f and g are continuous along A, then g is the continuation of f in D_2 and vice versa.

In our current setting, this theorem immediately leads to the Schwarz reflexion principle, which states that $f(\bar{z}) = \overline{f(z)}$ and holds for the whole imaginary plane except non-analytic part on the real axis above x_{th}.

Therefore, for $x > x_{\text{th}}$ and ε being an infinitesimal real constant, we have

$$\begin{aligned} f(x - i\varepsilon) &= \overline{f(x + i\varepsilon)} \\ &= f(x + i\varepsilon) - 2i\text{Im}f(x + i\varepsilon), \end{aligned} \tag{A.3}$$

which corresponds to Eq. (A.2). Thus we have finally obtained the dispersion relation of Eq. (3.2).

Next, let us consider the case, in which the integral Eq. (3.2) diverges. This problem can be fixed by using subtracted correlators as shown below. For instance, if the divergence is logarithmic, it suffices to employ the singly subtracted correlator:

$$\begin{aligned} \tilde{\Pi}(q^2) &\equiv \Pi(q^2) - \Pi(0) \\ &= \frac{q^2}{\pi} \int_0^\infty ds \frac{\text{Im}\Pi(s + i\varepsilon)}{s(s - q^2)}. \end{aligned} \tag{A.4}$$

In this way, one power of s can be included into the denominator, therefore making the integral convergent. This procedure can be repeated arbitrarily many times, by subtracting more and more terms from the Taylor expansion of $\Pi(q^2)$ around $q^2 = 0$, thus it is possible to cure divergences of any power. We however note, that by applying the Borel transformation, which contains infinitely many differentiations of q^2, all subtraction terms (which are polynomials in q^2) vanish and the integral of Eq. (3.2) is automatically turned into convergent one. Therefore, as long as the Borel transformation is applied, one usually does not have to worry about divergences and subtraction terms and can work directly with Eq. (3.2), the original form of the dispersion relation.

Appendix B
The Fock-Schwinger Gauge

In this appendix, the derivative expansions of the gluonic and quark fields, given in Eqs. (3.26) and (3.27), are derived. These expressions are valid only in the Fock-Schwinger gauge (Fock 1937; Schwinger 1954) (sometimes also referred to as the "fixed point gauge"),

$$(x - x_0)^\mu A^a_\mu(x) = 0, \tag{B.1}$$

where we set $x_0 = 0$ in the following. The derivation is based on the discussions given in Dubovikov and Smilga (1981) and Shifman (1980).

Multiplying $G^a_{\mu\nu}$, which is defined as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{B.2}$$

by x_μ and using Eq. (B.1), we get (with $x_0 = 0$)

$$x_\mu G^a_{\mu\nu}(x) = x^\mu \partial_\mu A^a_\nu(x) + A^a_\nu(x). \tag{B.3}$$

Then, x is replaced by αx after which we integrate by α:

$$\begin{aligned}\int_0^1 d\alpha \alpha x^\mu G^a_{\mu\nu}(\alpha x) &= \int_0^1 d\alpha \alpha \frac{d}{d\alpha} A^a_\nu(\alpha x) + \int_0^1 d\alpha A^a_\nu(\alpha x) \\ &= A^a_\nu(x).\end{aligned} \tag{B.4}$$

Here, the first line has been obtained by using

$$\frac{d}{d\alpha} A^a_\nu(\alpha x) = x^\mu \frac{\partial}{\partial(\alpha x^\mu)} A^a_\nu(\alpha x) = \frac{x^\mu}{\alpha} \partial x^\mu A^a_\nu(\alpha x). \tag{B.5}$$

Taylor expanding $G^a_{\mu\nu}(\alpha x)$ around $\alpha x = 0$ on the left hand side of Eq. (B.4) and carrying out the integration of α, we arrive at

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3, © Springer Japan 2013

$$A^a_\nu(x) = \frac{1}{2}x^\mu G^a_{\mu\nu}(0) + \frac{1}{3}x^\mu x^\alpha \partial_\alpha G^a_{\mu\nu}(0) + \frac{1}{8}x^\mu x^\alpha x^\beta \partial_\alpha \partial_\beta G^a_{\mu\nu}(0) + \cdots \quad \text{(B.6)}$$

For obtaining the final result, we have to show that the derivative ∂ can be replaced by the covariant derivative D in the above equation. The Fock-Schwinger gauge actually makes this substitution possible, as will be shown below.

The Taylor expansion of Eq. (B.1)

$$x^\mu A^a_\mu(x) = x^\mu \left[A^a_\mu(0) + x^\alpha \partial_\alpha A^a_\mu(0) + \frac{1}{2}x^\alpha x^\beta \partial_\alpha \partial\beta A^a_\mu(0) + \cdots \right] = 0, \quad \text{(B.7)}$$

which has to be valid for any value of x_μ, leads to the equations

$$\begin{aligned} x^\mu A^a_\mu(0) &= 0, \\ x^\mu x^\alpha \partial_\alpha A^a_\mu(0) &= 0, \\ &\cdots \end{aligned} \quad \text{(B.8)}$$

As the covariant derivative applied to gluon fields is defined as

$$D_\mu = \partial_\mu - igT^a A^a_\mu, \quad \text{(B.9)}$$

where T^a are the generators of $SU(3)$ in the adjoint representation and g is the strong coupling constant, we can derive the following relations:

$$\begin{aligned} x^\alpha \partial_\alpha G^a_{\mu\nu}(0) &= x^\alpha D_\alpha G^a_{\mu\nu}(0), \\ x^\alpha x^\beta \partial_\alpha \partial_\beta G^a_{\mu\nu}(0) &= x^\alpha x^\beta \partial_\alpha D_\beta G^a_{\mu\nu}(0) = x^\alpha x^\beta D_\alpha D_\beta G^a_{\mu\nu}(0), \\ &\cdots \end{aligned} \quad \text{(B.10)}$$

This shows that the derivatives can be substituted by the covariant derivatives, giving us thus the final result:

$$\begin{aligned} A^a_\mu(x) = {} & \frac{1}{2}x^\nu G^a_{\nu\mu}(0) + \frac{1}{3}x^\nu x^\alpha [D_\alpha G_{\nu\mu}(0)]^a \\ & + \frac{1}{8}x^\nu x^\alpha x^\beta [D_\alpha D_\beta G_{\nu\mu}(0)]^a + \cdots \end{aligned} \quad \text{(B.11)}$$

Next, we consider the quark fields. For this purpose, we simply Taylor expand the field $q(x)$ around $x = 0$, giving

$$q(x) = q(0) + x^\mu \partial_\mu q(0) + \frac{1}{2!}x^\nu x^\mu \partial_\nu \partial_\mu q(0) + \cdots \quad \text{(B.12)}$$

Now, the relations of Eq. (B.10) are valid also for the covariant derivative living in the fundamental representation,

$$D_\mu = \partial_\mu - ig\frac{\lambda^a}{2}A^a_\mu, \tag{B.13}$$

in which λ^a are the Gell-Mann matrices. Therefore, as for the gluonic fields above, we can simply interchange the derivatives of Eq. (B.12) with the covariant derivative, leading to the desired result:

$$q(x) = q(0) + x^\mu D_\mu q(0) + \frac{1}{2!}x^\nu x^\mu D_\nu D_\mu q(0) + \cdots \tag{B.14}$$

Appendix C
The Quark Propagator

To calculate the free quark propagator with no coupling to gluons and no long range correlations is most simple. It is given in standard textbooks of quantum field theory (such as Peskin and Schroeder 1995) and we here state only the result:

$$
\begin{aligned}
\langle 0_{\text{pert.}}|T[q(x)\overline{q}(0)]|0_{\text{pert.}}\rangle \equiv S_0(x) &= \int d^4p e^{-ipx} \frac{i}{\not{p} - m_q} \\
&\approx \int d^4p e^{-ipx} \left(\frac{i}{\not{p}} + \frac{im_q}{p^2} \right) + \mathscr{O}(m_q^2) \\
&\approx \frac{i}{2\pi^2} \frac{\not{x}}{x^4} - \frac{m_q}{4\pi^2} \frac{1}{x^2} + \mathscr{O}(m_q^2)
\end{aligned} \tag{C.1}
$$

$|0_{\text{pert.}}\rangle$ stands for the perturbative vacuum, where all condensates and expectation values of matter fields vanish. The last line of the above equation gives us the first two terms of Eq. (3.28).

C.1 Coupling with Gluon Fields

Here, the behavior of the quark propagator $S_A(x)$ in an external gluon field is discussed. Such a propagator satisfies the equation

$$
(i\not{\partial} + g\not{A} - m_q)S_A(x) = i\delta^4(x), \tag{C.2}
$$

and is expanded in powers of the external field $A_\mu^a(x)$. Expressed in Feynman diagrams, this expansion is shown in Fig. C.1, while mathematically it is given as

$$
\begin{aligned}
S_A(x) = S_0(x) &+ \int d^4y S_0(x-y) ig\not{A}(y) S_0(y) \\
&+ \int d^4y d^4z S_0(x-y) ig\not{A}(y) S_0(y-z) ig\not{A}(z) S_0(z) + \cdots
\end{aligned} \tag{C.3}
$$

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3, © Springer Japan 2013

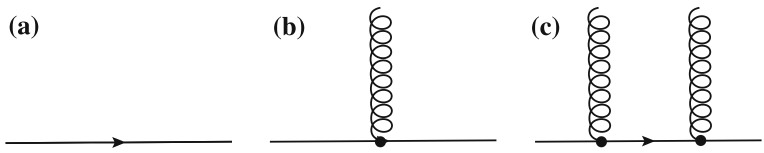

Fig. C.1 The first three diagrams of the external field expansion. **a** has already been calculated in Eq.(C.1). Here, we calculate **b** while **c** and the other higher order terms are neglected

Switching to the momentum representation and expressing the gluon field with the first term of Eq. (3.26), we get for the second term (Fig. C.1b):

$$
\begin{aligned}
&\int d^4y S_0(x-y) ig \not{A}(y) S_0(y) \\
&\approx \int d^4y \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{-ip(x-y)} e^{-iqy} \frac{i(\not{p}+m_q)}{p^2-m_q^2} ig\gamma^\mu \\
&\quad \times \left(\frac{y^\alpha}{2} G_{\alpha\mu}(0)\right) \frac{i(\not{q}+m_q)}{q^2-m_q^2} \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i(\not{p}+m_q)}{p^2-m_q^2} ig\gamma^\mu \left(\frac{i}{2} G_{\alpha\mu}(0)\right) \frac{\partial}{\partial q_\alpha} \left(\frac{i(\not{q}+m_q)}{q^2-m_q^2}\right)\Bigg|_{q=p} \\
&= -\frac{i}{4} g G^{\mu\nu}(0) \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{\sigma_{\mu\nu}(\not{p}+m_q)+(\not{p}+m_q)\sigma_{\mu\nu}}{(p^2-m_q^2)^2} \qquad \text{(C.4)} \\
&\approx -\frac{i}{4} g G^{\mu\nu}(0) \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{\sigma_{\mu\nu}\not{p}+\not{p}\sigma_{\mu\nu}}{p^4} \\
&\quad -\frac{i}{2} g m_q G^{\mu\nu}(0)\sigma_{\mu\nu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^4} + \mathcal{O}(m_q^2) \\
&= -\frac{i}{32\pi^2} g G^{\mu\nu}(0) \frac{\sigma_{\mu\nu}\not{x}+\not{x}\sigma_{\mu\nu}}{x^2} \\
&\quad -\frac{1}{32\pi^2} g m_q G^{\mu\nu}(0)\sigma_{\mu\nu} \ln\left(-\frac{x^2\Lambda^2}{4} + 2\gamma_{EM}\right) + \mathcal{O}(m_q^2).
\end{aligned}
$$

This results provides us with terms number three and four of Eq. (3.28).

C.2 Non-Perturbative Contributions

As a next step, we have to consider long range fluctuations of quarks and gluons in the quark propagator, which are expressed by various condensates such as $\langle\overline{q}q\rangle$, $\langle\overline{q}g\sigma Gq\rangle$ or $\langle\frac{\alpha_s}{\pi}G^2\rangle$.

For this, we make use of Eq. (3.27), substitute it into

$$\langle q_i^a(x)\overline{q}_j^b(0)\rangle, \tag{C.5}$$

and investigate its first few terms one after the other. For making the following manipulations easier and more tractable, we have here explicitly denoted the color and spinor indices as a, b and i, j, respectively. Note also that the non-perturbative components of the quark (and gluon) fields behave as classical fields that satisfy the equations of motion. Therefore, is valid to omit the time ordering operator $T[\dots]$.

For the first term, since it is sandwiched between the vacuum, only the scalar and color-singlet component survives. We can hence write

$$\langle q_i^a(0)\overline{q}_j^b(0)\rangle = A\delta^{ab}\delta_{ij}. \tag{C.6}$$

Taking the contractions of color and spinor indices on both sides and using the fact that quarks are Fermions and therefore anti-symmetric, we get

$$A = -\frac{1}{12}\langle\overline{q}q\rangle \tag{C.7}$$

for A.

The second term can be expressed as

$$x^\mu\left\langle\left[D_\mu q_i(0)\right]^a\overline{q}_j^b(0)\right\rangle = x^\mu\frac{\delta^{ab}}{3}\langle D_\mu q_i(0)\overline{q}_j(0)\rangle, \tag{C.8}$$

because, like above, only the color-singlet term survives. Then, the Dirac indices i, j are expanded with the complete set of $1, \gamma^5, \gamma_\nu, \gamma^5\gamma_\nu, \sigma_{\nu\rho}$, which gives

$$\begin{aligned}x^\mu\langle\left[D_\mu q_i(0)\right]^a\overline{q}_j^b(0)\rangle = x^\mu\frac{\delta^{ab}}{3}\Bigg(&-\frac{\delta_{ij}}{4}\langle\overline{q}(0)D_\mu q(0)\rangle - \frac{\gamma^5_{ij}}{4}\langle\overline{q}(0)\gamma^5 D_\mu q(0)\rangle\\ &-\frac{\gamma^\nu_{ij}}{4}\langle\overline{q}(0)\gamma_\nu D_\mu q(0)\rangle + \frac{(\gamma^5\gamma^\nu)_{ij}}{4}\langle\overline{q}(0)\gamma^5\gamma_\nu D_\mu q(0)\rangle\\ &-\frac{\sigma^{\nu\rho}_{ij}}{4}\langle\overline{q}(0)\sigma_{\nu\rho}D_\mu q(0)\rangle\Bigg).\end{aligned} \tag{C.9}$$

In this equation, only the third term can have the same quantum numbers as the vacuum and thus all the other terms vanish. Moreover, the scalar component of $\gamma_\nu D_\mu$ can be obtained as $\frac{g_{\nu\mu}}{4}\,\not{D}$, which leads to

$$x^\mu\langle\left[D_\mu q_i(0)\right]^a\overline{q}_j^b(0)\rangle = -x^\mu\frac{\delta^{ab}}{48}(\gamma_\mu)_{ij}\langle\overline{q}(0)\,\not{D}q(0)\rangle. \tag{C.10}$$

The equation of motion $\not{D}q = -im_q q$ is then used to derive the final result:

$$x^{\mu}\langle\big[D_{\mu}q_i(0)\big]^a\overline{q}_j^b(0)\rangle = \frac{im_q}{48}(\not{x})_{ij}\delta^{ab}\langle\overline{q}q\rangle. \tag{C.11}$$

Next, we consider the third term of Eq. (3.27). As before, the only the color singlet part has to be retained and therefore

$$\frac{1}{2}x^{\mu}x^{\nu}\langle\big[D_{\mu}D_{\nu}q_i(0)\big]^a\overline{q}_j^b(0)\rangle = \frac{\delta^{ab}}{6}x^{\mu}x^{\nu}\langle D_{\mu}D_{\nu}q_i(0)\overline{q}_j(0)\rangle. \tag{C.12}$$

Then, we do the same as in Eq. (C.9) and expand the Dirac indices. As is readily understood, only the component proportional to δ_{ij} survives because it is the only one containing a scalar part with positive parity. We thus get

$$\frac{1}{2}x^{\mu}x^{\nu}\langle\big[D_{\mu}D_{\nu}q_i(0)\big]^a\overline{q}_j^b(0)\rangle = -\frac{\delta_{ij}\delta^{ab}}{24}x^{\mu}x^{\nu}\langle\overline{q}(0)D_{\mu}D_{\nu}q(0)\rangle. \tag{C.13}$$

Finally, using the fact that the scalar part of $D_{\mu}D_{\nu}$ is $\frac{g_{\mu\nu}}{4}D^2$ and the equation $D^2 = \frac{1}{2}g\sigma Gq - m_q^2 q$, which is easily derived from the equation of motion, the following result is obtained:

$$\begin{aligned}\tfrac{1}{2}x^{\mu}x^{\nu}\left\langle\big[D_{\mu}D_{\nu}q_i(0)\big]^a\overline{q}_j^b(0)\right\rangle &= -\tfrac{x^2}{96}\delta_{ij}\delta^{ab}\langle\overline{q}(0)D^2q(0)\rangle \\ &\approx -\tfrac{x^2}{192}\delta_{ij}\delta^{ab}\langle\overline{q}g\sigma Gq\rangle + \mathcal{O}(m_q^2).\end{aligned} \tag{C.14}$$

The derivation of the fourth term and the fifth term of Eq. (3.27) is more involved, although the basic techniques are essentially the same. Here, only the results are stated:

$$\frac{1}{6}x^{\mu}x^{\nu}x^{\rho}\left\langle D_{\mu}D_{\nu}D_{\rho}q_i^a(0)\overline{q}_j^b(0)\right\rangle \approx \frac{im_q x^2}{2^7 3^2}(\not{x})_{ij}\delta^{ab}\langle\overline{q}g\sigma Gq\rangle + \mathcal{O}(m_q^2, g^2), \tag{C.15}$$

$$\begin{aligned}\tfrac{1}{24}x^{\mu}x^{\nu}x^{\rho}x^{\sigma}\left\langle D_{\mu}D_{\nu}D_{\rho}D_{\sigma}q_i^a(0)\overline{q}_j^b(0)\right\rangle \approx& -\tfrac{\pi^2 x^4}{2^8 3^3}\delta_{ij}\delta^{ab}\langle\overline{q}q\rangle\left\langle\tfrac{\alpha_s}{\pi}G^2\right\rangle \\ &+\mathcal{O}(m_q^2, g^2).\end{aligned} \tag{C.16}$$

The explicit derivation of Eq. (C.15) can be found in Chap. 6 of Ioffe et al. (2010). Furthermore, we note that for deriving Eq. (C.16), in addition to the method explained above, the vacuum saturation approximation has been assumed and the contraction formula for gluon fields

$$\langle G^k_{\mu\nu} G^l_{\rho\sigma} \rangle = \frac{\delta^{kl}}{2^5 3} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \langle G^2 \rangle \tag{C.17}$$

has been used.

Altogether, the results of this section give the remaining non-perturbative terms of Eq. (3.28).

Appendix D
Non-Perturbative Coupling of Quarks and Gluons

In this appendix, we derive the form of the non-perturbative coupling between quarks and gluons, given in Eq. (3.30). Our starting point is the following expression,

$$\begin{aligned}\big\langle 0|q_i^a(x)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\big\rangle &= \big\langle 0|q_i^a(0)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\big\rangle \\ &\quad + x^\alpha\big\langle 0|D_\alpha q_i^a(0)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\big\rangle + \cdots, \end{aligned} \tag{D.1}$$

for which we treat each of the two terms separately.

D.1 The First Term

In contrast to the calculations in the preceding appendix, one here has to build a color octet from the quarks, which is then contracted with the gluon for constructing an overall color-singlet operator. Therefore, for the first term, we get

$$\begin{aligned}\langle 0|q_i^a(0)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\rangle &= -2\left(\frac{\lambda^l}{2}\right)^{ab}\left\langle 0|\overline{q}_j(0)\left(\frac{\lambda^l}{2}\right)gG^k_{\mu\nu}(0)q_i(0)|0\right\rangle \\ &= -\frac{1}{4}\left(\frac{\lambda^k}{2}\right)^{ab}\langle 0|\overline{q}_j(0)gG_{\mu\nu}(0)q_i(0)|0\rangle, \end{aligned} \tag{D.2}$$

after which the spinor indices are expanded as in Eq. (C.9). We then obtain

$$\begin{aligned}\big\langle 0|q_i^a(0)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\big\rangle &= -\frac{(\sigma_{\rho\sigma})_{ij}}{32}\left(\frac{\lambda^k}{2}\right)^{ab}\big\langle 0|\overline{q}(0)\sigma^{\rho\sigma}gG_{\mu\nu}(0)q(0)|0\big\rangle \\ &= -\frac{(\sigma_{\mu\nu})_{ij}}{2^6 3}\left(\frac{\lambda^k}{2}\right)^{ab}\langle\overline{q}g\sigma Gq\rangle, \end{aligned} \tag{D.3}$$

which is the final result.

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3,

D.2 The Second Term

The second term of Eq. (D.1) can be calculated in the same way, it is however somewhat more complicated. First, we construct a color octet from the quark fields and combine it with the octet from the gluon, as before:

$$\begin{aligned}
&x^\alpha \langle 0|D_\alpha q_i^a(0) g G^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\rangle \\
&= -2\left(\frac{\lambda^l}{2}\right)^{ab} x^\alpha \langle 0|\overline{q}_j(0)\left(\frac{\lambda^l}{2}\right) g G^k_{\mu\nu}(0) D_\alpha q_i(0)|0\rangle \\
&= -\frac{1}{4}\left(\frac{\lambda^k}{2}\right)^{ab} x^\alpha \langle 0|\overline{q}_j(0) g G_{\mu\nu}(0) D_\alpha q_i(0)|0\rangle .
\end{aligned} \tag{D.4}$$

Next, we expand the spinor indices of the quark fields. Doing this, it is clear that only the terms with γ_μ or $\gamma_5\gamma_\mu$ can survive, because from all other terms it is not possible to construct a scalar operator. Thus we have

$$\begin{aligned}
\langle 0|\overline{q}_j(0) g G_{\mu\nu}(0) D_\alpha q_i(0)|0\rangle = &\frac{(\gamma^\beta)_{ij}}{4}\langle 0|\overline{q}(0)\gamma_\beta g G_{\mu\nu}(0) D_\alpha q(0)|0\rangle \\
&-\frac{(\gamma_5\gamma^\beta)_{ij}}{4}\langle 0|\overline{q}(0)\gamma_5\gamma_\beta g G_{\mu\nu}(0) D_\alpha q(0)|0\rangle .
\end{aligned} \tag{D.5}$$

Subsequently, we expand the remaining parts into their possible Lorentz structures. Parity considerations tell us that the first term can only be proportional to $g_{\mu\nu}g_{\alpha\beta}$, $g_{\mu\alpha}g_{\nu\beta}$ or $g_{\mu\beta}g_{\alpha\nu}$ and the second term only to $\varepsilon_{\mu\nu\alpha\beta}$, giving

$$\langle 0|\overline{q}(0)\gamma_\beta g G_{\mu\nu}(0) D_\alpha q(0)|0\rangle = A g_{\mu\nu}g_{\alpha\beta} + B g_{\mu\alpha}g_{\nu\beta} + C g_{\mu\beta}g_{\alpha\nu}, \tag{D.6}$$

and

$$\langle 0|\overline{q}(0)\gamma_5\gamma_\beta g G_{\mu\nu}(0) D_\alpha q(0)|0\rangle = D\varepsilon_{\mu\nu\alpha\beta}. \tag{D.7}$$

Contracting Eq. (D.6) with $g^{\mu\nu}g^{\alpha\beta}$, $g^{\mu\alpha}g^{\nu\beta}$ and $g^{\mu\beta}g^{\alpha\nu}$, we get three equations, which lead to

$$\begin{aligned}
A &= 0, \\
B &= -\frac{1}{12}\langle 0|\overline{q}(0)\gamma^\rho g G_{\rho\sigma}(0) D^\sigma q(0)|0\rangle , \\
C &= \frac{1}{12}\langle 0|\overline{q}(0)\gamma^\rho g G_{\rho\sigma}(0) D^\sigma q(0)|0\rangle .
\end{aligned} \tag{D.8}$$

On the other hand, contracting Eq. (D.7) with $\varepsilon^{\mu\nu\alpha\beta}$, we obtain

$$D = \frac{1}{24}\langle 0|\overline{q}(0)\gamma_5\varepsilon^{\mu\nu\alpha\beta}\gamma_\beta g G_{\mu\nu}(0) D_\alpha q(0)|0\rangle , \tag{D.9}$$

which can be rearranged by Eq. (E.21) of Appendix E and the equation of motion for quarks ($\not{D}q = -im_q q$). This gives

$$D = \frac{i}{12}\langle 0|\overline{q}(0)\gamma^\rho gG_{\rho\sigma}(0)D^\sigma q(0)|0\rangle + \frac{im_q}{24}\langle 0|\overline{q}(0)g\sigma G(0)q(0)|0\rangle. \tag{D.10}$$

Now, all we need is an expression of $\langle 0|\overline{q}(0)\gamma^\rho gG_{\rho\sigma}(0)D^\sigma q(0)|0\rangle$ in form of known condensates. The details of the manipulations necessary for this task are explained in Chap. 6 of Ioffe et al. (2010) and we here give only the final result:

$$\begin{aligned}&\langle 0|\overline{q}(0)\gamma^\rho gG_{\rho\sigma}(0)D^\sigma q(0)|0\rangle\\ =&-\frac{1}{2}\langle 0|\overline{q}(0)\gamma^\rho\left(\frac{\lambda^n}{2}\right)q(0)g\big[D^\sigma G_{\rho\sigma}(0)\big]^n|0\rangle\\ &-\frac{m_q}{2}\langle 0|\overline{q}(0)g\sigma G(0)q(0)|0\rangle.\end{aligned} \tag{D.11}$$

The first term on the right hand side of the above equation can be rewritten using the equation of motion for gluons. This gives a term proportional to g^2, which we neglect here. If one wants to calculate higher orders of α_s, it however has to be retained.

Assembling the results of Eqs. (D.6)–(D.11) we finally get

$$\langle 0|\overline{q}_j(0)gG_{\mu\nu}(0)D_\alpha q_i(0)|0\rangle = \frac{m_q}{2^5 3}\big[(\gamma_\nu)_{ij}g_{\alpha\mu} - (\gamma_\mu)_{ij}g_{\alpha\nu}\big]\langle 0|\overline{q}g\sigma Gq|0\rangle, \tag{D.12}$$

which leads to

$$\begin{aligned}&x^\alpha\langle 0|D_\alpha q_i^a(0)gG^k_{\mu\nu}(0)\overline{q}^b_j(0)|0\rangle\\ \approx&-\frac{m_q}{2^7 3}\left(\frac{\lambda^k}{2}\right)^{ab}[(\gamma_\nu)_{ij}x_\mu - (\gamma_\mu)_{ij}x_\nu]\langle 0|\overline{q}g\sigma Gq|0\rangle + \mathcal{O}(g^2).\end{aligned} \tag{D.13}$$

The spinor part of this result $(\gamma_\nu x_\mu - \gamma_\mu x_\nu)$ can be further manipulated according to formula of Eqs. (E.21) and (E.22) in Appendix E as follows

$$\begin{aligned}\gamma_\nu x_\mu - \gamma_\mu x_\nu &= -x^\lambda(g_{\nu\lambda}\gamma_\mu - g_{\mu\lambda}\gamma_\nu)\\ &= -x^\lambda(i\varepsilon_{\mu\nu\lambda\rho}\gamma_5\gamma^\rho - g_{\mu\nu}\gamma_\lambda + \gamma_\mu\gamma_\nu\gamma_\lambda)\\ &= -\frac{i}{2}(\not{x}\sigma_{\mu\nu} + \sigma_{\mu\nu}\not{x}) + i\sigma_{\mu\nu}\not{x}\\ &= -\frac{i}{2}(\not{x}\sigma_{\mu\nu} - \sigma_{\mu\nu}\not{x}),\end{aligned} \tag{D.14}$$

which gives

$$x^\alpha \langle 0|D_\alpha q_i^a(0) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle \approx \frac{im_q}{2^8 3}(\not{x}\sigma^{\mu\nu} - \sigma^{\mu\nu}\not{x})_{ij}\left(\frac{\lambda^k}{2}\right)^{ab}\langle\overline{q}g\sigma Gq\rangle + \mathscr{O}(g^2). \tag{D.15}$$

The final form of the non-perturbative coupling of quarks and gluons is then

$$\begin{aligned}\langle 0|q_i^a(x) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle \approx &-\frac{(\sigma_{\mu\nu})_{ij}}{2^6 3}\left(\frac{\lambda^k}{2}\right)^{ab}\langle\overline{q}g\sigma Gq\rangle \\ &+ \frac{im_q}{2^8 3}(\not{x}\sigma^{\mu\nu} - \sigma^{\mu\nu}\not{x})_{ij}\left(\frac{\lambda^k}{2}\right)^{ab}\langle\overline{q}g\sigma Gq\rangle \\ &+ \mathscr{O}(m_q^2, g^2),\end{aligned} \tag{D.16}$$

which corresponds to Eq. (3.30) of the main text.

Appendix E
Gamma Matrix Algebra

When doing calculations in the QCD sum rule technique, various properties of gamma matrices are frequently used. A few of the most convenient formulae concerning these gamma matrices are given in this appendix. Note that we here use the convention $\varepsilon^{0123} = 1$ for the totally antisymmetric Levi-Civita tensor.

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{E.1}$$

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{E.2}$$

$$\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{E.3}$$

$$\gamma_\mu \gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu} \tag{E.4}$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{E.5}$$

$$C \equiv i\gamma^2\gamma^0 \text{ (charge conjugation matrix)} \tag{E.6}$$

$$C = C^* = -C^\dagger = -C^T = -C^{-1} \tag{E.7}$$

$$C^2 = -1 \tag{E.8}$$

$$C\gamma_5 = \gamma_5 C \tag{E.9}$$

$$C\Gamma^T C = +\Gamma \quad \text{for} \quad \Gamma = \gamma_\mu,\ \sigma_{\mu\nu},\ \gamma_5\sigma_{\mu\nu} \tag{E.10}$$

$$C\Gamma^T C = -\Gamma \quad \text{for} \quad \Gamma = \gamma_5,\ \gamma_5\gamma_\mu,\ (\not{x}\sigma^{\mu\nu} + \sigma^{\mu\nu}\ \not{x}) \tag{E.11}$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = -2\gamma_\nu \tag{E.12}$$

$$\gamma^\mu \gamma_\alpha \gamma_\beta \gamma_\mu = 4g_{\alpha\beta} \tag{E.13}$$

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3, © Springer Japan 2013

$$\gamma^\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\mu = -2\gamma_\gamma \gamma_\beta \gamma_\alpha \tag{E.14}$$

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = 12 \tag{E.15}$$

$$\sigma^{\alpha\beta}\gamma^\mu\gamma^\nu\sigma_{\alpha\beta} = 4\gamma^\nu\gamma^\mu + 8g^{\mu\nu} = 16g^{\mu\nu} - 4\gamma^\mu\gamma^\nu \tag{E.16}$$

$$\sigma^{\alpha\beta}(\text{odd number of } \gamma\text{-matrices})\sigma_{\alpha\beta} = 0 \tag{E.17}$$

$$(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})(\not{x}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\not{x}) = 24x^2 \tag{E.18}$$

$$(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})\gamma^\mu(\not{x}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\not{x}) = 8(x^2\gamma^\mu + 2x^\mu\not{x}) \tag{E.19}$$

$$\begin{aligned}(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})\gamma^\mu\gamma^\nu(\not{x}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\not{x}) = \\ 8(4x^2g^{\mu\nu} - 2x^\mu\gamma^\nu\not{x} + 2x^\nu\gamma^\mu\not{x} - x^2\gamma^\mu\gamma^\nu)\end{aligned} \tag{E.20}$$

$$\varepsilon_{\mu\nu\lambda\rho}\gamma^\rho = -i\gamma_5(g_{\mu\nu}\gamma_\lambda - g_{\mu\lambda}\gamma_\nu + g_{\nu\lambda}\gamma_\mu - \gamma_\mu\gamma_\nu\gamma_\lambda) \tag{E.21}$$

$$\not{x}\sigma_{\mu\nu} + \sigma_{\mu\nu}\not{x} = -2\varepsilon_{\mu\nu\alpha\beta}\gamma_5\gamma^\alpha x^\beta \tag{E.22}$$

$$\sigma_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\lambda}\gamma_5\sigma^{\rho\lambda} \tag{E.23}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu} \tag{E.24}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda] = 4(g_{\mu\nu}g_{\rho\lambda} - g_{\mu\rho}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\rho}) \tag{E.25}$$

$$\mathrm{Tr}[\text{odd number of } \gamma\text{-matrices}] = 0 \tag{E.26}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\cdots] = \mathrm{Tr}[\cdots\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu] \tag{E.27}$$

$$\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu] = 0 \tag{E.28}$$

$$\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = -4i\varepsilon_{\mu\nu\rho\sigma} \tag{E.29}$$

$$\mathrm{Tr}[\gamma_\mu\not{x}\gamma_\nu\not{x}] = 8x_\mu x_\nu - 4x^2 g_{\mu\nu} \tag{E.30}$$

$$\begin{aligned}\mathrm{Tr}[\gamma_\mu(\not{x}\sigma_{\rho\lambda} + \sigma_{\rho\lambda}\not{x})\gamma_\nu\not{x}] &= -8ix^2(g_{\rho\mu}g_{\lambda\nu} - g_{\rho\nu}g_{\lambda\mu}) \\ &\quad + 8ix_\mu(x_\rho g_{\lambda\nu} - x_\lambda g_{\rho\nu}) \\ &\quad + 8ix_\nu(x_\lambda g_{\rho\mu} - x_\rho g_{\lambda\mu})\end{aligned} \tag{E.31}$$

$$\mathrm{Tr}[\sigma_{\rho\lambda}\gamma_5\not{x}\gamma_\mu] = 4\varepsilon_{\alpha\rho\lambda\mu}x^\alpha \tag{E.32}$$

$$\mathrm{Tr}[\sigma_{\rho\lambda}\not{x}\gamma_\mu] = 4i(g_{\rho\mu}x_\lambda - g_{\lambda\mu}x_\rho) \tag{E.33}$$

$$\mathrm{Tr}[\sigma_{\mu\nu}\sigma_{\rho\lambda}] = 4(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \tag{E.34}$$

$$\mathrm{Tr}[(\not{x}\sigma_{\mu\nu} + \sigma_{\mu\nu}\not{x})(\not{x}\sigma_{\rho\lambda} + \sigma_{\rho\lambda}\not{x})] = -16\varepsilon_{\sigma\mu\nu\alpha}\varepsilon^{\sigma}{}_{\rho\lambda\beta}x^\alpha x^\beta \tag{E.35}$$

$$\varepsilon_{\sigma\mu\nu\alpha}\varepsilon^{\sigma}{}_{\rho\lambda\beta} = g_{\mu\rho}g_{\nu\lambda}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\rho}g_{\alpha\lambda} + g_{\mu\lambda}g_{\nu\beta}g_{\alpha\rho} - g_{\nu\rho}g_{\mu\lambda}g_{\alpha\beta} - g_{\nu\beta}g_{\mu\rho}g_{\alpha\lambda} - g_{\nu\lambda}g_{\mu\beta}g_{\alpha\rho} \tag{E.36}$$

$$\varepsilon_{\alpha\beta\mu\nu}\varepsilon^{\alpha\beta}{}_{\rho\lambda} = 2(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \tag{E.37}$$

Appendix F
The Fourier Transformation

When QCD sum rules of hadrons containing light quarks are considered, one usually carries out the OPE in coordinate space and Fourier transforms the result back into momentum space at the end of the calculation. We give in this appendix the necessary formulae for this task.

F.1 The Standard Case

For the standard Fourier transformation, one can derive (almost) all formulae needed in practical calculations from

$$\int d^4x e^{iqx} \frac{1}{(x^2)^n} = i(-1)^n \frac{2^{4-2n}\pi^2}{\Gamma(n-1)\Gamma(n)} (q^2)^{n-2} \ln(-q^2) + P_{n-2}(q^2), \quad \text{(F.1)}$$

which is valid for $n \geq 2$. The derivation of this equation can be found in Novikov et al. (1984). $P_m(q^2)$ stands for a polynomial of q^2 of order m. The coefficients of this polynomial are in fact divergent, but as they will in any case vanish when the Borel transform is applied, we omit them in the following discussion.

Variations of Eq. (F.1) with various tensor structures can be constructed by taking appropriate derivatives:

$$\int d^4x e^{iqx} \frac{x^\mu x^\nu \dots}{(x^2)^n} = \left(\frac{\partial}{i\partial q_\mu}\right)\left(\frac{\partial}{i\partial q_\nu}\right) \dots \int d^4x e^{iqx} \frac{1}{(x^2)^n}. \quad \text{(F.2)}$$

F.2 The “Old Fashioned” Case

Here, we evaluate the Fourier transforms of the various terms occurring in the “old fashioned” correlator of Eq. (3.53).

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3, © Springer Japan 2013

F.2.1 Dimension 0–5 Terms

For the terms appearing at dimensions 0–5, it is most convenient to work in coordinate space. Therefore, we directly use the expressions of Eq. (3.61) and substitute them into Eq. (3.53). For the dimension 5 term, this gives

$$\int d^4x\theta(x_0)e^{iqx}\frac{i}{(x^2-i\varepsilon)^2}=$$
$$\int dx_0\theta(x_0)e^{iq_0x_0}\int d^3\mathbf{x}\frac{i}{(x_0^2-\mathbf{x}^2-i\varepsilon)^2}e^{-i\mathbf{q}\cdot\mathbf{x}}. \tag{F.3}$$

First, we calculate the integrals over the spacial angles θ and ϕ, leading to

$$\frac{2\pi}{|\mathbf{q}|}\int dx_0\theta(x_0)e^{iq_0x_0}\int_{-\infty}^{\infty}dr\frac{r}{(r-x_0+i\varepsilon)^2(r+x_0-i\varepsilon)^2}e^{i|\mathbf{q}|r}, \tag{F.4}$$

where we have used the definition $r\equiv|\mathbf{x}|$. Next comes the integral over r, which can be done in a standard way with the help of the Cauchy theorem. We thus obtain

$$\pi^2\int dx_0\theta(x_0)\frac{1}{x_0-i\varepsilon}e^{ix_0(q_0-|\mathbf{q}|)}. \tag{F.5}$$

At this point, we can drop $|\mathbf{q}|$, as there is no danger that the limit $|\mathbf{q}|\to 0$ leads to a divergence. Furthermore, we here introduce the Fourier transformed expression for the Heaviside step function:

$$\theta(x_0)=\frac{1}{2\pi i}\int dk_0\frac{1}{k_0-i\varepsilon}e^{ix_0k_0}. \tag{F.6}$$

We then get

$$\frac{\pi}{2i}\int dk_0\int dx_0\frac{1}{k_0-i\varepsilon}\frac{1}{x_0-i\varepsilon}e^{ix_0(q_0+k_0)}. \tag{F.7}$$

Making use of Eq. (F.6) now for the integral over x_0, giving the final result:

$$\pi^2\int_{-q_0}^{\infty}dk_0\frac{1}{k_0-i\varepsilon}=-\pi^2\ln(-q_0-i\varepsilon)+\pi^2\ln(\infty-i\varepsilon). \tag{F.8}$$

Here, we encounter a divergence in the second term, which, however, leads to no relevant contribution to the imaginary part of the correlator, which is the only quantity that is needed for the sum rules. We can therefore ignore it and hence have obtained the result used in Eq. (3.63).

The term of dimension 0, 3 and 4 can be calculated in a similar fashion. The main difference is that due to the larger powers in the denominator, the poles used in the Cauchy theorem leading to Eq. (F.5) are of a larger degree, which however does not introduce any essential new difficulties. We here give only the results:

$$\begin{aligned}
\text{dim. }0: &\int d^4x\theta(x_0)e^{iqx}\frac{\not{x}}{(x^2-i\varepsilon)^5} = -\frac{\pi^2}{2^9 3}\gamma_0 q_0^5 \ln(-q_0-i\varepsilon),\\
\text{dim. }3: &\int d^4x\theta(x_0)e^{iqx}\frac{i}{(x^2-i\varepsilon)^3} = \frac{\pi^2}{8} q_0^2 \ln(-q_0-i\varepsilon),\\
\text{dim. }4: &\int d^4x\theta(x_0)e^{iqx}\frac{\not{x}}{(x^2-i\varepsilon)^3} = -\frac{\pi^2}{4}\gamma_0 q_0 \ln(-q_0-i\varepsilon).
\end{aligned} \tag{F.9}$$

We here, as above, have taken the limit $|\mathbf{q}| \to 0$. Note, however that in all the above calculations, this limit can be taken only *after* the integral over r has been carried out, as otherwise the factor $1/|\mathbf{q}|$ appearing in Eq. (F.4) can not be properly treated.

F.2.2 Dimension 6–10 Terms

For the terms with dimension larger than 5, the calculation is simpler if one starts from momentum space. This means that we take the expressions of Eq. (3.61) to substitute them into Eq. (3.53). In fact, the basic steps of the calculation are already given in Eqs. (3.54) and (3.55) of the main text and the result of the dimension 6 term can be directly deduced from these equations by setting $m^{\pm} = 0$. For the sake of illustration, we here show the calculation of one more term, the one of dimension 7.

For getting the result of this term we have to evaluate the following integral:

$$\int d^4x\theta(x_0)e^{iqx}\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{p^2+i\varepsilon}. \tag{F.10}$$

Here, we first employ the expression of Eq. (F.6) and perform the integral over x. This yields

$$\frac{1}{2\pi i}\int dk_0 \frac{1}{k_0-i\varepsilon}\frac{1}{(k_0+q_0)^2-\mathbf{q}^2+i\varepsilon}. \tag{F.11}$$

For calculating the remaining integral over k_0, we note that there are three poles in the integrand, two in the upper half of the imaginary plane, and one in the lower half. Closing thus the contour in the lower half of the imaginary plane, we pick up the residue of this single pole, which originates from the second factor of the integrand. We then obtain the final result as

$$\frac{1}{2}\frac{1}{\sqrt{\mathbf{q}^2}-i\varepsilon}\frac{1}{q_0-\sqrt{\mathbf{q}^2}+i\varepsilon}. \tag{F.12}$$

As is clear from this expression, we can at this point not take the limit $|\mathbf{q}| \to 0$ as it would lead to a divergence. This problem is only cured after the integral over q_0 is carried out as shown in Eq. (3.64).

The Fourier transforms of the higher order terms can be calculated analogously and we here show only the results.

$$\begin{aligned}
\text{dim. } 8: &\int d^4x\theta(x_0)e^{iqx}\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{\not{p}}{(p^2+i\varepsilon)^2} \\
&= \gamma_0 \frac{1}{4}\frac{1}{\sqrt{\mathbf{q}^2}-i\varepsilon}\frac{1}{(q_0-\sqrt{\mathbf{q}^2}+i\varepsilon)^2}, \\
\text{dim. } 9: &\int d^4x\theta(x_0)e^{iqx}\int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{1}{(p^2+i\varepsilon)^2} \\
&= \frac{1}{4}\left[\frac{1}{(\sqrt{\mathbf{q}^2}-i\varepsilon)^2}\frac{1}{(q_0-\sqrt{\mathbf{q}^2}+i\varepsilon)^2} - \frac{1}{(\sqrt{\mathbf{q}^2}-i\varepsilon)^3}\frac{1}{q_0-\sqrt{\mathbf{q}^2}+i\varepsilon}\right].
\end{aligned} \tag{F.13}$$

Note that the result of the dimension 8 term in principle also contains expressions proportional to $\mathbf{q}\cdot\gamma$ as long as the $|\mathbf{q}| \to 0$ limit is not taken. These however vanish when the traces of Eq. (3.56) are taken and are therefore of no relevance here.

F.2.3 Evenness (Oddness) of Dimension 6, 8,…(7, 9,…) Terms

In this section, we proof the statement made in the main text, that the imaginary parts of the terms corresponding to dimensions $6, 8, \ldots$ $(7, 9, \ldots)$ in Eq. (3.63) are even (odd) functions of q_0, if one takes the limit $|\mathbf{q}| \to 0$.

First, by following the same steps that lead from Eq. (F.10) to Eq. (F.11), and setting $|\mathbf{q}| = 0$, we notice that all terms appearing at dimensions $6, 8, \ldots$ can generally be written down as

$$\begin{aligned}
&\frac{1}{2\pi i}\int dk_0 \frac{1}{k_0-i\varepsilon}\frac{k_0+q_0}{[(k_0+q_0)^2+i\varepsilon]^n} \equiv F_1(q_0) \\
&= \frac{1}{2\pi i}\int dk_0 \frac{1}{k_0-i\varepsilon}\frac{k_0+q_0}{(k_0+q_0+i\varepsilon)^n(k_0+q_0-i\varepsilon)^n}. \qquad (n=1,2,\ldots)
\end{aligned} \tag{F.14}$$

Here, we are ignoring any proportional real constant, including γ_0. Similarly, for dimensions $7, 9, \ldots,$ we get

$$\begin{aligned}
&\tfrac{1}{2\pi i}\int dk_0 \frac{1}{k_0-i\varepsilon}\frac{1}{[(k_0+q_0)^2+i\varepsilon]^n} \equiv F_2(q_0) \\
&= \tfrac{1}{2\pi i}\int dk_0 \frac{1}{k_0-i\varepsilon}\frac{1}{(k_0+q_0+i\varepsilon)^n(k_0+q_0-i\varepsilon)^n}. \qquad (n=1,2,\ldots)
\end{aligned} \tag{F.15}$$

Next, we take the imaginary parts and, after some simple manipulations, get for $F_1(q_0)$

$$
\begin{aligned}
\mathrm{Im}F_1(q_0) &= \frac{1}{2i}\Big[F_1(q_0) - \overline{F_1(q_0)}\Big] \\
&= -\frac{1}{4\pi}\int dk_0\Big(\frac{1}{k_0 - i\varepsilon} + \frac{1}{k_0 - i\varepsilon}\Big)\Bigg[\frac{k_0 + q_0}{(k_0 + q_0 + i\varepsilon)^n (k_0 + q_0 - i\varepsilon)^n} \\
&\qquad - \frac{-k_0 + q_0}{(-k_0 + q_0 + i\varepsilon)^n(-k_0 + q_0 - i\varepsilon)^n}\Bigg],
\end{aligned}
\tag{F.16}
$$

while the result for $F_2(q_0)$ is

$$
\begin{aligned}
\mathrm{Im}F_2(q_0) &= \frac{1}{2i}\Big[F_2(q_0) - \overline{F_2(q_0)}\Big] \\
&= -\frac{1}{4\pi}\int dk_0\Big(\frac{1}{k_0 - i\varepsilon}\Big)\Bigg[\frac{1}{(k_0 + q_0 + i\varepsilon)^n (k_0 + q_0 - i\varepsilon)^n} \\
&\qquad - \frac{1}{(-k_0 + q_0 + i\varepsilon)^n(-k_0 + q_0 - i\varepsilon)^n}\Bigg].
\end{aligned}
\tag{F.17}
$$

Having the above equations at hand, it is now a trivial matter to show that

$$
\mathrm{Im}F_1(-q_0) = \mathrm{Im}F_1(q_0), \tag{F.18}
$$

and

$$
\mathrm{Im}F_2(-q_0) = -\mathrm{Im}F_2(q_0), \tag{F.19}
$$

which proofs our statement made in the main text, that the imaginary parts of the terms of dimensions 6, 8, ... (7, 9, ...) in the OPE of the "old fashioned" correlator are even (odd) functions of q_0 in the limit $|\mathbf{q}| \to 0$.

Appendix G
Derivation of the Shannon-Jaynes Entropy

In this appendix, we will provide two derivations for the Shannon-Jaynes entropy, given in Eq. (4.7), the first one making use of the law of large numbers, the second one being an axiomatic construction based on locality, system independence and scaling. We will mainly follow the explanations given in Asakawa et al. (2001).

G.1 Proof Based on the Law of Large Numbers

The proof of the Shannon-Jaynes entropy can be given by the so-called "monkey argument", which basically assumes that the probability of the spectral function $\rho(\omega)$ follows a certain Poisson distribution, as will be explained below.

What we need to derive is the probability of $\rho(\omega)$ to be in a specific region V of its allowed phase space. Formally, this probability can be denoted as

$$P(\rho \in V) = \frac{1}{Z(\alpha)} \int_V [d\rho] W(\alpha S(\rho)), \tag{G.1}$$

where $Z(\alpha)$ is simply a normalization constant, while α is just an arbitrary parameter, whose significance will be discussed in the main text. Furthermore, $S(\rho)$ is the entropy that we want to derive here. Also note that, as $P(\rho \in V)$ should have the maximum value where $S(\rho)$ is largest, the function W should be a monotone increasing function.

According to the monkey argument, we now divide the function $\rho(\omega)$ into N ω-regions of the same size and consider a monkey that throws M balls into them. The throwing process is not completely arbitrary, but is assumed to follow a certain pattern. Thus, each region has a probability p_i $(1 \leq i \leq N)$ to receive a ball, leading to an expectation value for the number of balls of $\lambda_i = Mp_i$. Furthermore, we denote the actual number of balls that reaches a specific region as n_i. From probability theory, we know that if we take M to be very large and keep λ_i fixed, the probability of n_i to have a certain value will behave according to a Poisson distribution. Therefore, the

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3,

probability for a certain combination of $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ to take place, can be written down as

$$P(\mathbf{n}) = \prod_{i=1}^{N} \frac{\lambda_i^{n_i} e^{-\lambda_i}}{n_i!}. \tag{G.2}$$

Here, the components of $\mathbf{n}$ are integers, and hence can not yet be considered to be a useful parametrization of the smooth function $\rho(\omega)$. We thus introduce a parameter q, with which we can make $\mathbf{n}$ proportional to the function $\rho(\omega)$:

$$\rho_i = q n_i, \tag{G.3}$$

where ρ_i stands for the value of $\rho(\omega)$ in the ith region of ω. Similarly, we can define the default model as

$$m_i = q\lambda_i. \tag{G.4}$$

We are now in a position to explicitly evaluate the probability of Eq. (G.1) as follows:

$$\begin{aligned} P(\rho \in V) &= \textstyle\sum_{\mathbf{n}\in V} P(\mathbf{n}) \simeq \frac{1}{q^N} \prod_{i=1}^{N} \int_V d\rho_i \frac{\lambda_i^{n_i} e^{-\lambda_i}}{n_i!} \\ &\simeq \textstyle\frac{1}{(2\pi q)^{N/2}} \int_V \prod_{i=1}^{N} \frac{\rho_i}{\sqrt{\rho_i}} e^{S(\rho)/q}, \end{aligned} \tag{G.5}$$

where we have used the Stirling approximation $n! \simeq \sqrt{2\pi n} e^{n\log n - n}$ in the last line. $S(\rho)$ is given as

$$S(\rho) = \sum_{i=1}^{N} \Big[\rho_i - m_i - \rho_i \log\big(\rho_i/m_i\big)\Big], \tag{G.6}$$

which is equivalent to Eq. (4.7) of the main text. Furthermore, it is seen from Eqs. (G.1) and (G.5) that $q = 1/\alpha$ and that the measure $[d\rho]$ and the normalization constant $Z(\alpha)$ can be expressed as

$$[d\rho] = \prod_{i=1}^{N} \frac{\rho_i}{\sqrt{\rho_i}}, \qquad Z(\alpha) = \left(\frac{2\pi}{\alpha}\right)^{N/2}. \tag{G.7}$$

As a last point, we also observe that the function W of Eq. (G.1) is a simple exponential and therefore indeed a monotone increasing function as it should be.

G.2 Proof Based on an Axiomatic Construction

For illustration, we here give another proof for the Shannon-Jaynes entropy, which is based on the four axioms of locality, coordinate invariance, system independence and scaling. This proof is less intuitive than the one given in the last section, it is, however, in some sense, more general, as it does not rely on the assumption of the Poisson distribution used in Eq. (G.2).

As in the last section, our task is to define a real functional, which satisfies the following condition:

$$\text{If } \rho_1 \text{ is a more probable function than } \rho_2\text{, then: } S(\rho_1) > S(\rho_2). \tag{G.8}$$

Thus, the most probable of all functions can be found by looking for a stationary point in $S(f)$, which follows from the equation

$$\delta_\rho S(\rho) = 0. \tag{G.9}$$

Let us now derive the actual form of $S(\rho)$ by considering the four axioms mentioned above.

G.2.0.1 Locality

This axiom declares that the values of $\rho(\omega)$ at various values of ω should independently contribute to $S(\rho)$ without any correlation. Therefore, one can conclude that $S(\rho)$ should be a local function of $\rho(\omega)$ and can written down as

$$S(\rho) = \int d\omega m(\omega)\phi(\rho(\omega), \omega), \tag{G.10}$$

where $m(\omega)$ can be considered to be the integration measure and must be positive definite. Furthermore, ϕ is an arbitrary function of $\rho(\omega)$ and ω, but cannot contain any derivatives of $\rho(\omega)$, as they would lead to correlations between different values of ω.

G.2.0.2 Coordinate Invariance

The axiom of coordinate invariance demands that $S(\rho)$ does not depend on what sort of coordinates one uses for the function $\rho(\omega)$. In other words, $S(\rho)$ should be invariant under the coordinate transformation $\omega' = \omega'(\omega)$. Now, using $\rho(\omega)d\omega = \rho'(\omega')d\omega'$ and $m(\omega)d\omega = m'(\omega')d\omega'$, one can understand that the right hand side of Eq. (G.10) can only be invariant if the function $\rho(\omega)$ appears in ϕ divided by $m(\omega)$, because of the relation $\rho(\omega)/m(\omega) = \rho'(\omega')/m'(\omega')$. Hence, we can express Eq. (G.10) as

$$S(\rho) = \int d\omega m(\omega)\phi(\rho(\omega)/m(\omega)). \tag{G.11}$$

G.2.0.3 System Independence

This axiom states that in case of a function $\rho_c(\omega_1, \omega_2)$ having two independent variables ω_1 and ω_2, this function can be written as a product of two functions: $\rho_c(\omega_1, \omega_2) = \rho_1(\omega_1)\rho_2(\omega_2)$. Moreover, the corresponding integration measure can be divided in the same way: $m_c(\omega_1, \omega_2) = m_1(\omega_1)m_2(\omega_2)$. A further consequence of the axiom is that the variance of $S(\rho_c)$ w.r.t. $\rho_c(\omega_1, \omega_2)$ is given as

$$\frac{\delta S(\rho_c)}{\delta \rho_c(\omega_1, \omega_2)} = \alpha(\omega_1) + \beta(\omega_2), \tag{G.12}$$

where $\alpha(\omega_1)$ and $\beta(\omega_2)$ are functions related to the variance of $S(\rho_c)$ w.r.t. $\rho_1(\omega_1)$ and $\rho_2(\omega_2)$, respectively.

Now, using the form for $S(\rho_c)$ that we have obtained in Eq. (G.11), we can write down $S(\rho_c)$ as

$$S(\rho_c) = \int d\omega_1 \int d\omega_2 m_c(\omega_1, \omega_2)\phi(\rho_c(\omega_1, \omega_2)/m_c(\omega_1, \omega_2)), \tag{G.13}$$

which gives

$$\frac{\delta S(\rho_c)}{\delta \rho_c(\omega_1, \omega_2)} = \frac{d\phi}{dZ}\bigg|_{Z=\rho_c(\omega_1,\omega_2)/m_c(\omega_1,\omega_2)} \equiv \sigma(Z = \rho_c(\omega_1, \omega_2)/m_c(\omega_1, \omega_2)). \tag{G.14}$$

Next, we act with $\partial^2/\partial\omega_1\partial\omega_2$ on the right hand sides of both Eqs. (G.12) and (G.14). As these should be equal, we are lead to the following equation for $\sigma(Z)$:

$$Z\frac{d^2\sigma(Z)}{dZ^2} + \frac{d\sigma(Z)}{dZ} = 0. \tag{G.15}$$

The above equation can be easily solved, giving $\sigma(Z) = c_1 \log(Z) + c_2$, from which we finally get the functional form of $\phi(Z)$ as

$$\phi(Z) = c_1 Z \log(Z) + (c_2 - c_1)Z + c_3, \tag{G.16}$$

where c_1, c_2 and c_3 are integration constants that are not yet determined at the current stage. Substituting the result of Eq. (G.16) into Eq. (G.11), $S(f)$ can now be given as

$$S(\rho) = \int d\omega\Big[c_1\rho(\omega)\log\left(\frac{\rho(\omega)}{m(\omega)}\right) + (c_2 - c_1)\rho(\omega) + c_3 m(\omega)\Big]. \tag{G.17}$$

Using this equation, we get $\delta^2/\delta\rho^2 S(\rho) = c_1/\rho$ and thus observe that the sign of c_1 completely determines the curvature of $S(\rho)$, as ρ is a positive definite function. Therefore, in order for $S(\rho)$ to be bounded from above, one has to chose c_1 to be negative.

G.2.0.4 Scaling

According to this axiom, in case of no additional information available on $\rho(\omega)$ (for instance, from the likelihood function of Eq. (4.4)), the most probable form of $\rho(\omega)$ should be equal to the integration measure $m(\omega)$. Thus the maximum of $S(\rho)$ should be at $\rho(\omega) = m(\omega)$.

The maximum of $S(\rho)$ of Eq. (G.17) can be obtained from the solution of $\frac{\delta S(\rho)}{\delta \rho} = 0$, which gives

$$\rho(\omega) = m(\omega)e^{-c_2/c_1}. \tag{G.18}$$

From this result, we can immediately conclude that for satisfying the scaling axiom, we need to set $c_2 = 0$. Thus, the form of $S(\rho)$ is now

$$S(\rho) = -c_1 \int d\omega \left[\rho(\omega) - \rho(\omega) \log\left(\frac{\rho(\omega)}{m(\omega)}\right) - \frac{c_3}{c_1} m(\omega) \right]. \tag{G.19}$$

As a last task, we still have to determine c_1 and c_3. Considering first c_3, we see from the above equation that the term proportional to this constant does not depend on $\rho(\omega)$ and is therefore not of much relevance in the present discussion. In order for $S(\rho)$ to vanish when $\rho(\omega)$ equals $m(\omega)$, one usually chooses $c_3 = c_1$ for convenience. As for c_1, we have already mentioned above that it should have a negative value. As can be observed from Eq. (G.19), its magnitude just becomes an overall normalization factor in front of the integral over ω, which can be arbitrarily chosen. Usually, one takes $c_1 = -1$ for simplicity. We thus are lead to

$$S(\rho) = \int d\omega \left[\rho(\omega) - m(\omega) - \rho(\omega) \log\left(\frac{\rho(\omega)}{m(\omega)}\right) \right], \tag{G.20}$$

which is indeed the Shannon-Jaynes entropy of Eq. (4.7).

Appendix H
Uniqueness of the Maximum of $P[\rho|GH]$

It is important for the MEM procedure that there is only one solution for $\rho(\omega)$, which maximizes the conditional probability $P[\rho|GH]$. We will proof in this short appendix, that the solution is indeed unique if it exists, following the discussion given in Asakawa et al. (2001).

For proofing the uniqueness of the solution for $\rho(\omega)$, we first have to show the correctness of the following mathematical statement:

> Given a real and smooth function $F(x_1, x_2, \ldots, x_n)$ with real variables $(x_1, x_2, \ldots, x_n) \in \mathbf{R^n}$, for which the matrix $\partial^2 F/\partial x_i \partial x_j$ is negative definite, the solution of the equations $\partial F/\partial x_i = 0$ is unique if it exists.

Note here that the negative definiteness of $\partial F/\partial x_i \partial x_j$ can be denoted as

$$\sum_{i,j=1}^{n} y_i \frac{\partial^2 F}{\partial x_i \partial x_j} y_j < 0 \qquad ({}^\forall y_i \in \mathbf{R}/\{\mathbf{0}\}). \tag{H.1}$$

For showing the above statement, we assume that there are two solutions for $\partial F/\partial x_i = 0$, $\mathbf{x_1}$ and $\mathbf{x_2}$, and define $\mathbf{x}(t) \equiv \mathbf{x_1} + t(\mathbf{x_2} - \mathbf{x_1})$ and $G(t) \equiv F(\mathbf{x}(t))$. From these definitions, we can immediately see that $dG(t)/dt$ satisfies

$$\left.\frac{dG(t)}{dt}\right|_{t=0} = \left.\frac{dG(t)}{dt}\right|_{t=1} = 0. \tag{H.2}$$

Now, from the smoothness of F, the function $G(t)$ must be continuous and differentiable. We can therefore use Rolle's theorem, which states that between $t = 0$ and $t = 1$, there must be at least one t which satisfies

$$\frac{d^2 G(t)}{dt^2} = \sum_{i,j=1}^{n} y_i \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{x}(t)} y_j = 0, \tag{H.3}$$

P. Gubler, *A Bayesian Analysis of QCD Sum Rules*, Springer Theses,
DOI: 10.1007/978-4-431-54318-3, © Springer Japan 2013

which leads to a contradiction with Eq. (H.1). Therefore, the solution of $\partial F/\partial x_i = 0$ must be unique if it exists.

Thus, all we have to do for proofing the uniqueness of the solution for $\rho(\omega)$, is to show that $Q(\rho)$ of Eq. (4.9) satisfies the condition analog to Eq. (H.1). Using the (discretized forms of) the likelihood function and the prior probability of Eqs. (4.4) and (4.7), we can derive

$$\sum_{i,j=1}^{N_\omega} y_i \frac{\partial^2 Q}{\partial\rho_i \partial\rho_j} y_j = -\frac{\alpha}{\Delta\omega}\sum_{i=1}^{N_\omega}\frac{y_i^2}{\rho_i} - \frac{\Delta x}{x_{\max}-x_{\min}}\sum_{j=1}^{N_x}\sum_{i=1}^{N_\omega}\frac{\left[K(x_j,\rho_i)y_i\right]^2}{\sigma^2(x_j)}. \quad \text{(H.4)}$$

Here, ρ_i represents the discretized data points of $\rho(\omega)$: $\rho_i \equiv \rho(\omega_i)\Delta\omega$, with $\Delta\omega \equiv \frac{\omega_{\max}-\omega_{\min}}{N_\omega}$ and $\omega_i \equiv \frac{i}{N_\omega}(\omega_{\max}-\omega_{\min}) + \omega_{\min}$. Similarly, x_j stands for $x_j \equiv \frac{j}{N_x}(x_{\max}-x_{\min}) + x_{\min}$ and Δx for $\Delta x \equiv \frac{x_{\max}-x_{\min}}{N_x}$. While the second term in principle can become 0 for certain values of $\mathbf{y}$, the first term is always negative because of $0 < \alpha$ and $0 \le \rho_i$. Therefore, we can conclude that

$$\sum_{i,j=1}^{N_\omega} y_i \frac{\partial^2 Q}{\partial\rho_i \partial\rho_j} y_j < 0 \qquad ({}^\forall y_i \in \mathbf{R}/\{\mathbf{0}\}), \quad \text{(H.5)}$$

which, together with the statement shown above, proofs the uniqueness of the maximum of $P[\rho|GH]$ if it exists.

References

Arfken G (1970) Mathematical methods for physicists, 2nd edn. Academic Press, New York
Asakawa M, Hatsuda T, Nakahara Y (2001) Prog Part Nucl Phys 46:459
Dubovikov MS, Smilga AV (1981) Nucl Phys B 185:109
Fock VA (1937) Sowj Phys 12:404
Ioffe BL, Fadin VS, Lipatov LN (2010) Quantum chromodynamics, perturbative and nonperturbative aspects. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, Cambridge
Morse PM, Feshbach H (1953) Methods of theoretical physics. McGraw-Hill Book Company, New York
Novikov VA, Shifman MA, Vainshtein AI, Zakharov VI (1984) Fortsch Phys 32:585
Peskin ME, Schroeder DV (1995) An introduction to quantum field theory. Westview Press, Boulder
Schwinger J (1954) Phys Rev 82:664
Shifman MA (1980) Nucl Phys B 173:13