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**A Brief Introduction
to Numerical Analysis**

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Preface

Probably I ought to explain why one more book on numerical methods can be useful. Without any doubt, there are many quite good and excellent books on the subject. But I know definitely that I did not realize this when I was a student. In this book, my first desire was to present those lectures that I wished I would have heard when I was a student.

Besides, in spite of the profusion of textbooks, introductory courses, and monographs on numerical methods, some of them are too elementary, some are too difficult, some are far too overwhelmed with applications, and most of them are too lengthy for those who want to see the whole picture in a short time.

I hope that the brevity of the course left me no chance to obscure the beauty and depth of mathematical ideas behind the theory and methods of numerical analysis.

I am convinced that such a book should be very concise indeed. It should be thoroughly structured, giving information in short sections which, ideally, are a half-page in length. Equally important, the book should not give an impression that nothing is left to work on in this field. Any time it becomes possible to say something about modern development and recent results, I do try to find time and place for this.

Still, I do not promise *easy* reading. This book is addressed, first, to those who study mathematics. Despite this, it is written so that it can be read by students majoring in physics and mathematics, and I believe it can be useful for advanced readers and researchers providing them with new findings and a new vision of the basic framework.

Somebody might remark that there is no excuse for brevity in the list of references at the end of this book. I could only agree and beg not to be blamed for this. I included in the list only the books that I felt influenced me most directly. Several imposing papers are also mentioned in the footnotes.

The book contains, in fact, a concise and closed exposition of the lectures given by the author to the 2–3 year students of the Chair of Mathematical Modelling of Physical Processes of the Faculty of Problems of Physics and Energetics of the Moscow Institute of Physics and Technology.

To conclude the preface, I get to its main purpose, to express my thanks. Above all, I am grateful to V. V. Voevodin, my first teacher who had inspired me by his way of doing science. His advice and encouragement were always of great importance to me.

Special thanks go to S. A. Goreinov and N. L. Zamarashkin. They were the first readers and found many opportunities to share with me their remarks and impressions.

It is my pleasure also to express my gratitude to G. I. Marchuk for suggesting these lectures.

December 1996

Eugene Tyrtysnikov

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