A Budget of Trisections

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Introduction

> My opinion of mankind is founded upon the mournful fact that, so far as I can see, they find within themselves the means of believing in a thousand times as much as there is to believe in. (Augustus De Morgan, *A Budget of Paradoxes*, Volume 1, page 115.)

In 1955, the mathematics library at the Carnegie Institute of Technology was in a room at the end of a long, long hall. For much of its length, the hall sloped slightly downward, either to follow the slope of the land underneath or to show what engineers and architects could do if they set their minds to it. The library was in a former classroom; it had three rows of shelves and four tables. Borrowers of books were on their honor to write on a card the name of a book taken out and the date it was taken. They were also on their honor to return it within two weeks. Now the Carnegie Institute of Technology is the Carnegie-Mellon University, and its mathematics library no doubt occupies much more space and has many more books. I suspect that the honor system is no longer in use.

Then, more than 30 years ago, the library was a pleasant place to be in the later afternoon, with classes over for the day and the setting sun coming in the two west-facing windows, lighting up the particles of dust in the air. Then as now, students of mathematics (I was a student of mathematics then) generally did not read anything that they were not required to read, so I was usually alone there, with no one to tell me to take my feet off the table, to ask whether I had some homework to do, or to suggest that I should leave the library and get out where the people are. I liked, and still do, to have my feet on a table, I was talented enough then not to have to struggle with my homework as much as most of my fellow students of mathematics, and, then as now, books were good company.

The library had a small and miscellaneous collection. Looking back, I see that it may have been intended as a student library, with the research library for the faculty located elsewhere, since there were hardly any of the rows and rows of bound periodicals so necessary for the researcher. On the other hand, there may have been no research library, for why would a student library have more than 40 volumes of the *Collected Works* of Euler, mostly paperbound and mostly in Latin? In any event, on the bottom shelf on the right of the first row of shelves were two volumes, bound in red and black, that one day caught my eye: *A Budget of Paradoxes* they were called, written by Augustus De Morgan. Paradoxes—those were "proofs" that $2 = 1$, achieved by dividing by zero; or that every triangle is isosceles, done by drawing a diagram incorrectly; or that $1 = -1$, accomplished by forgetting that square roots are never negative: how could there be two large volumes devoted to them? I picked up the first volume and found that it was not like that at all. It had some very strange things in it, written about in a style that I had never before read. The first edition was published in London in 1872, and the contents were contributions, to various periodicals, together with additions, that De Morgan had made over the years before his death in 1871.

De Morgan was a mathematician, a teacher of mathematics, a prolific writer, and an original man. Here is a portion of his nonmathematical biography, by J. M. Dubbey, in the *Dictionary of Scientific Biography*:

> De Morgan, Augustus (*b.* Madura, Madras presidency, India, June 1806; *d.* London, England, 18 March 1871), *mathematics.*
>
> De Morgan's father was a colonel in the Indian Army; and his mother was a friend of Abraham de Moivre, and granddaughter of James Dodson, author of the *Mathematical Canon.* At the age of seven months De Morgan was brought to England, where his family settled first at Worcester and then at Taunton. He attended a succession of private schools at which he acquired a mastery of Latin, Greek, and Hebrew and a strong interest in mathematics before the age of fourteen. He also acquired an intense dislike for cramming, examinations, and orthodox theology.
>
> De Morgan entered Trinity College, Cambridge, in February 1823 and placed first in the first-class division in his senior year; he was disappointed, however, to graduate only as fourth wrangler in 1827. After contemplating a career in either medicine or law, De Morgan successfully applied for the chair of mathematics at the newly formed University College, London, in 1828 on the strong recommendation of his former tutors, who included Airy and Peacock. When, in 1831, the college council dismissed the professor of anatomy without giving reasons, he immediately resigned on principle. He resumed in 1836, on the accidental death of his successor, and remained there until a second resignation in 1866.
>
> De Morgan's life was characterized by powerful religious convictions. While admitting a personal faith in Jesus Christ, he abhorred any suspicion of hypocrisy or sectarianism and on these grounds refused an M. A., a fellowship at Cambridge, and ordination. In 1837 he married Sophia Elizabeth Frend, who wrote his biography in 1882. De Morgan was never wealthy; and his researches into all branches of knowledge, together with his prolific output of writing, left little time for social or family life. However, he was well known for his humor, range of knowledge, and sweetness of disposition.

De Morgan wrote influential texts, his pupils included the well-known mathematicians Sylvester and Todhunter, he invented the term *mathematical induction*, he did fundamental work in logic, and he wrote *Arithmetical Books*, which was "probably the first significant work of scientific bibliography."

> De Morgan's peripheral mathematical interests included ... a curious work entitled *Budget of Paradoxes*, which considers, among other things, the work of would-be circle squarers.

The library's copy of *A Budget of Paradoxes* was an edition published in Chicago, in 1915. It had very extensive notes added by David Eugene Smith, a historian of mathematics, because, he said,

> Hundreds of names are referred to in the text that were more or less known in England half a century ago, but are now forgotten and were never familiar elsewhere. Many books that were then current have now passed out of memory, and much that agitated England in De Morgan's time seems now like ancient history.

Smith provided a short paragraph for *every* name mentioned, and sometimes the notes filled half the page. They constituted an impressive feat of scholarship and were often as fascinating as the text.

The text was a curious mixture. Most of it I could understand, but there were parts which I could not penetrate on first or second reading. With repeated efforts, I have now mastered them all. De Morgan wrote with unfailing clarity, but a separation of 5,000 miles and 100 years can make things hard to see. *A Budget of Paradoxes* is full of wit, and it is full of high seriousness too; De Morgan is sometimes exasperated, sometimes genial, sometimes very earnest. By *paradox*, De Morgan meant anything counter to general opinion, and he defined a *paradoxer* as one who puts forth paradox. Today, acupuncture (for one example of many possible) is a paradox, and those who write booklets with diagrams of the body indicating where the needles should be stuck are paradoxers. Paradox is something not orthodox, so paradoxers are not necessarily wrong in what they maintain. Not necessarily, but usually they are. The *Budget* deals mostly with paradoxes in mathematics, physics, and religion, but it also contains digressions, sometimes quite long, on topics of amazing variety. There are anagrams of *Augustus De Morgan*—"Great gun! Do us a sum!" is one (*Budget*, Vol. 1, p. 138)—an Astronomer's Drinking Song:

> Copernicus, that learned wight
>   the glory of his nation,
> with draughts of wine refreshed his sight,
>   and saw the earth's rotation;

> Each planet then its orb described
> the moon got under way, sir;
> these truths from nature he imbibed
> for he drank his bottle a day, sir!
> (*Budget*, Vol. 1, p. 380.)

and so on—many book reviews, such as

> *The Decimal System as a Whole.* By Dover Statter. London and Liverpool, 1856.
>
> The proposition is to make everything decimal. The day, now 24 hours, is to be made 10 hours. The year is to have ten months, Unsuber, Duober, etc. Fortunately there are ten commandments, so there will be neither addition to, nor deduction from, the moral law. But the twelve apostles! Even rejecting Judas, there is a whole apostle of difficulty. These points the author does not touch. (*Budget*, Vol. 2, p. 80.)

And on and on. I read on and on, even through the parts I could not comprehend. In 1954, Dover Publications had reprinted the 1915 edition and a department store in Pittsburgh stocked a copy. Even though it was terrifically expensive ($4.95, or 15 percent of my total weekly expenditures for food and everything else), I bought it. I am glad that I did, and it is five feet away from me as I write this.

Sometimes a book can give you the illusion that you know its author, that you know what he was like and how he behaved. Some biographies can do the same thing—I *know* Gauss—although other personalities are forever beyond reach—no one will *ever* know Newton. It is easy to become acquainted with Augustus De Morgan. It is no surprise to read in his wife's biography of him

> After we were settled at No. 41 Chalcot Villas, Adelaide Road (at that time nearly surrounded by fields, and fully two miles from the College), he left the house always before eight o'clock in the morning, and met the omnibus in the Hampstead Road, which took him to Grafton Street a short time before the lecture began. He returned to dinner at five o'clock; and as he only gave himself about half an hour's rest after dinner before going to his library, where he wrote or read for four or five hours, he seldom gave up an evening to friends without feeling that his work for the next day had accumulated.

Nor is it a surprise to read a letter from De Morgan to a friend, in 1869,

> You think, one letter of yours says, that I am feeling the effects of hard work; in fact, that I have been working too hard. Rid your mind of the idea. I have never been *hard* working, but I have been very *continuously* at work. I have never *sought* relaxation. And why? Because it would have killed me. Amusement is real hard work to me. To relax is to forage about the books with no particular object, and not bound to go with anything.

Yes, that is exactly the author of *A Budget of Paradoxes.*

A paradoxer could be anyone from Galileo, with his fearfully unorthodox ideas about the motion of the earth, to someone with a trisection of the angle with straightedge and compass alone. Some things which were once paradox are now orthodox: when Wegener advanced his idea of continental drift early in this century he got not acceptance but laughs and sneers. Today, if anyone doubts the truth of plate tectonics the doubter gets the laughs and sneers. There is, however, no chance whatever that status of the angle trisector will ever change. Such people, paradoxers who are demonstrably wrong, I will call *cranks.* I will apply the same term to people who maintain positions which, though not demonstratably wrong, have in almost everyone's opinion a very, very low probability of being correct. It would be impossible to *prove* that the Pyramidologists—that group, not as large as it once was, that asserts that the builders of the Great Pyramid of Cheops incorporated into it information about the future of the race, discoverable by taking measurements—are wrong, but cranks is what they are. Believers in the sunken continent of Atlantis, in the prophesies of Nostradamus, or in the existence of unidentified flying objects piloted by small green aliens, followers of Dianetics, members of the Flat Earth Society, mediums, and those who patronize them—cranks all, and all deserving of the title. *A Budget of Paradoxes* contains quite a few cranks. There were many circle-squarers, that is, people who thought that they could construct a square with exactly the same area as a circle using straightedge and compass alone. And a pencil, of course. This amounts to the same thing as determining the value of π, the ratio of the circumference of a circle to its diameter. Many of De Morgan's circle-squarers claimed that the value of π was a rational number, such as the popular $3\frac{1}{8}$. It had been known since 1761, to mathematicians though not to paradoxers, that such a thing was impossible. It had been proved to be impossible, and that was the end of it. Those circle-squarers were cranks. Other circle-squarers gave constructions which led to irrational values of π, but it was not until 1877 that it was finally proved that no straightedge and compass construction could ever succeed in squaring the circle. Until then, there was a hope that a construction could be found, but those circle-squarers with constructions which led to wrong values of π were cranks too.

Most of the mathematical cranks in *A Budget of Paradoxes* are circle-squarers. Circle-squaring was very popular in the nineteenth century. Circles were squared in books, pamphlets, single sheets, and even newspapers. Nowadays very few people square circles. There are trisections in newspapers now and then, but no circle-squarings. I like to think that the *Budget* was the main cause in the decline. As De Morgan says,

> If I had before me a fly and an elephant, having never seen more than one such magnitude of either kind; and if the fly were to endeavor to persuade me that he was larger than the elephant, I might possibly be placed in a difficulty. The apparently little creature might use such arguments about the effect of distance, and might appeal to such laws of sight and hearing as I, if unlearned in those

> things, might be wholly unable to reject. But if there were a thousand flies, all buzzing, to appearance, about the great creature; and, to a fly, declaring, each one for himself, that he was bigger than the quadruped; and all giving different and frequently contradictory reasons; and each one despising and opposing the reasons of the others—I should feel quite at my ease. I should certainly say, My little friends, the case of each of you is destroyed by the rest. I intend to show flies in the swarm, with a few larger animals, for reasons to be given. (*Budget*, Vol. 1, p. 1.)

There once was a swarm of circle-squarers, but there is one no longer. Perhaps De Morgan caused that, and I hope that this book will have a similar effect on angle trisectors.

For whatever reason—I know what it is, but I need not give it here—De Morgan and cranks gripped me and I have been gripped ever since. Whenever I came across a piece of crank literature, I held onto it. I began to try to search it out. I put an advertisement in *Fate* magazine, at the time a sort of *Reader's Digest* of the occult. I went to the Library of Congress and copied all that I could find. I wrote to 600 departments of mathematics, asking whether anyone there were a fellow collector or, if not, whether they had any material on file. *No* one would admit to collecting it. It was hard to believe: in a country where there are collectors of barbed wire and of telephone insulators, is there no one collecting crank mathematics? Evidently not, though many departments were generous in sending material. But many other departments had none at all. Some departments throw such stuff away immediately. Other departments file all of their correspondence, and that from cranks goes into a folder labeled *Nuts*, *Crackpots*, or something similar, though that label is not fair. What De Morgan wrote is still true:

> They are in all ranks and occupations, of all ages and characters. They are very earnest people, and their purpose is *bona fide* the dissemination of their paradoxes. A great many—the mass, indeed—are illiterate, and a great many waste their means and are in or approaching penury. (*Budget*, Vol. 1, p. 8.)

except that the illiteracy rate is lower now. Or perhaps De Morgan was counting the writer of the following letter (I have received similar ones) illiterate:

> When a Gentleman of your standing in Society Clad with those honors Can not understand or Solve a problem That is explicitly explained by words and Letters and mathematically operated by figures He had best consult the wise proverd
>
> Do that which thou Canst understand and Comprehend for thy good.
>
> I would recommend that Such Gentleman Change his business
>
> And appropriate his time and attention to a Sunday School to Learn what he Could and keep the Little Children from durting their Close.

With Sincere feelings of Gratitude for your weakness and Inability I am,

Sir your superior in Mathematics (*Budget*, Vol. 2, pp. 16–17.)

The trouble is that every department sooner or later purges its files and the crackpot folder is always destroyed. So this material, the folk mathematics of the time, is lost forever. It is a shame, for crank mathematics is worth at least as much attention as many things to which scholars pay attention. If, reader, you know of any mathematical crank literature, I would be pleased to have it, or a copy of it: I have the collector's lust. I want it *all*.

Crank mathematics is mostly produced by amateurs, doing it for fun, or for the challenge, or for the fictitious million dollars they have heard is the reward for solving some problem. There are a few cases, some notorious in the mathematical community, of professional mathematicians turning into cranks (or displaying their crankhood, if you believe as I do that cranks are born and not made), but not many. You might think that all angle trisecting is done by amateurs since professionals understand what "impossible" means in mathematics, but there is an exception; in the Budget of Trisections which is the last section of this book, there appears a trisector with a Ph.D. degree in mathematics who earned his living teaching mathematics. It is incredible that such a person should be a trisector, but there he is. But almost all crank mathematics is done by amateurs: somewhere, sometime, a person remembers that time in tenth-grade geometry when the teacher said that it is impossible to trisect angles with straightedge and compass alone. Then, not knowing what "impossible" means in mathematics, he gets out compass and straightedge and starts to attack the problem. He is on his way. It is fun. He finds a construction that looks very good for an angle of 30°, giving 10° so closely that not even a big protractor can tell the difference, but which doesn't look quite right for a 75° angle. How to modify the construction so that the trisection point will be a little more in that direction? Maybe if I draw *this* circle instead of *that* one. It is fun: problems presented, problems solved, small triumphs and small failures, all together with the sensual pleasure of drawing neat diagrams with sharpened pencils. The feel of a nicely balanced sharp-pointed compass! The crisp intersection of line and line! The tangent line, grazing the circle at precisely one point! What better way to spend an evening?

There are many better ways. The fun stops when the construction is completed and then the frustration begins. Mathematicians will not look at the construction. Or they look at it and use trigonometry, of which the average trisector is ignorant, to show the trisector that he is wrong. Or they give him proofs, which he cannot understand, that the trisection is impossible. All of them, over and over, say he is wrong, wrong, wrong. What was pleasure turns to pain. It is too bad. The purpose of this book is to reduce the amount of that pain. If a trisector, or a potential trisector, can see the swarm in the Budget of Trisections, perhaps he will turn from the trisection to some other form of recreational mathematics. That will be pure gain, both for him and for the mathematical community. I have the opinion, based on no evidence whatsoever, that there is a crank personality and that some people are destined to be cranks, so perhaps he will become a physics crank, refuting Einstein; no gain for him, but gain for the mathematical community, which will no longer be bothered with him. Of course, if he starts to square the circle, no one has gained, but we can only try our best.

By the way, the trisection *is* impossible. It was proved so by Wantzel, in 1837. That is all the typical history of mathematics has to say about Wantzel, if it mentions him at all. He deserves more, especially since he also gave the first proof of a result about regular polygons that the great Gauss said that he had proved but that he had never published. Pierre Laurent Wantzel was, as his first names show, French; he was born in 1814; he was a talented mathematician; he published his proof in Liouville's *Journal de Math.* (volume 2, 1837, pages 366–372); and he died in his thirty-fifth year. A contemporary wrote

> Ordinarily he worked evenings, not lying down until late; he then read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death.

Let us give thanks, those of us who lack it, that we were not cursed with ambition.

The idea of Wantzel's proof is that constructions with straightedge and compass allow one to perform with lines and circles the arithmetical operations of addition, subtraction, multiplication, division, taking a square root, and no others, whereas to trisect an angle it is necessary, in effect, to take a cube root. There is no way of combining any number of plusses, minuses, timeses, divides, and square roots to get a cube root, and that is why the trisection cannot be done with straightedge and compass alone. The proof uses nothing more advanced than algebra, but it is just hard enough and just involved enough to make it impossible for anyone without a good deal of mathematical training to understand it. Even so, well-meaning but not clear-thinking professors of mathematics continue to send copies of the proof to trisectors. I have not included the proof here because if you have the background necessary to understand it, you know already that the trisection is impossible and you do not need to read it, and if you lack the background, it would be too hard for you to understand. Besides, it can be found in many places: Courant and Robbins's *What Is Mathematics?*, likely to be in any good library, and in many other places, less accessible but nevertheless numerous. Test the quality of the department of mathematics of the nearest college by asking them for references. They will be pleased. People seldom ask departments of mathematics anything.

Hardly any mathematical training is necessary to read this book. There is a little trigonometry here and there, but it may be safely skipped. There are hardly any equations. There are no exercises at the end of the sections, and there will be no final examination. The worst victim of mathematics anxiety can read this book with profit and dry palms. It is suitable for giving as a present.

There is hardly any literature on the trisection. Every now and then a journal will print an approximate trisection, and the proof that it is impossible appears here and there, but the only book I know of devoted to it is *The Trisection Problem*, by R. C. Yates, a slim volume mostly devoted to the proof of impossibility and to constructions using compass, straightedge, and something else to accomplish the trisection. Trisectors are mentioned hardly at all.

What follows, then, is something which has never been done before: it is an effort to do something which may be as impossible as trisecting the angle: namely to put an end to trisections and trisectors. It was inspired by *A Budget of Paradoxes* but lacks its wide scope, and its author attempts the manner of De Morgan but lacks his vast erudition. There were giants in those days.

It is customary in introductions to give thanks where due. The list of those who have sent me material is too long to print, I may accidentally leave someone out, and some mathematicians may object to being in the same list with trisectors, so I will do the same thing an author of a paper does to acknowledge assistance: write

> The author thanks the referee for numerous helpful suggestions.

or something similar; only the editor and the referee know who is meant. So I will just say, to all of you who ever wrote me a letter, thanks. Thanks very much. You know who you are.

It would be absurd to suppose that there are no errors in what follows. I would be glad to hear of any.

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